Dynamic Learning in Supply Chain with Repeated Transactions and Service Contributions

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Abstract

This research investigates how the experience learned in repeated transactions by consumers and suppliers would affect supply-chain partners stratigic decisions such as price, order quantity and service level. The supply-chain focused in this study includes two manufacturers producing competing products and selling them through a common retailer. The consumer demand depends on two factors: (1) retailer price and (2) service level provided by the manufacturer in the past and current transaction periods. The Game theory is used to understand the interactions between the horizontal competiton among two suppliers and their vertical interactions to the common retailer in the two-period looking ahead decision environment. The dynamic system concept is integrated to the game-theoretic model for understanding the evolution of the stratigic decisions over multiple time perods. Our research shows that if demand is only sensitive to price in its learning process, the supplier with any type of cost advantage will be able to support more services to its customers and thus capture a larger market to gain more profit than its competitor. Comparison of our model to the myopic model indicates that myopic suppliers are not capable to cope with the learning consumers. Their market sizes shrink and they earn less profit over time. On the other hand, the supplier who uses the learning experience to plan future investment can prevent this phenomenon from happening and enhance their competitiveness.

KEY WORDS: Learning; Horizontal Strategic Interaction; Manufacturer Service;
Retail Pricing; Supply Chain Management; Vertical Interaction;

1 Introduction

Experience learned from past transactions can help consumer decide what brand of products to buy and what price he/she is willing to pay. This experience can also help supplier decide their future investment for improving its competitiveness. This research focuses on experience learned from supply-chain partners' repeated transactions for deciding product price, order quantity and service level, and also from consumers' memory of price and service difference of product brands. In this research, service is defined as any action that the manufacturer takes to "help the customers obtain maximum value from their purchases" (Goffin 1999). Example of services include post-sale customer support, product advertising, improved product quality, product delivery, etc. To limit the scope of research, the supply-chain considered has two manufacturers producing competing products and selling them through a common retailer.

It is only recently that "learning" through repeated transactions has been integrated into multi-period models. There are two streams of research on "learning" in the literature. Petruzzi and Dada (2001, 2002) [17], [18] and Cachon and Porteus (1999) [2] are among the studies in the first group which regard "learning" as a process of updating information on demand distribution. For instance, Petruzzi and Dada (2001) analyzed inventory and pricing decisions in a two-period retail setting when an opportunity to refine information about uncertain demand is available. Specifically, they determined the optimal stocking and pricing policies over time when a given market parameter of the demand process, though fixed, is initially unknown. Petruzzi and Dada (2002) [18] extended the problem by considering a multiple-period problem. The authors use dynamic programming techniques to formulate their optimization model.

Another stream of research embeds "learning" into the demand function as part of demand modelling. Vilcassim *et al.* (1999) [25] used this approach in their analysis of price and advertising competition among firms in a given product market. Firm (or brand) level demand functions account for the contemporaneous and carry-over effects of these marketing activities, and also allow for the effects of competitor actions. This approach enables them to quantify both the direction and magnitude of competitive reactions, and also to identify the form of market conduct that generates the particular pattern of interaction. Our research follow the latter approach of "learning" to study the repeated transactions problems.

Our study approaches the problem by introducing a new methodology by integrating the game theory with dynamic system concept in understanding the behavior (and impact) of horizontal and vertical firms' decision making process in multiple periods. By applying this new methodology, we answer the following questions: (*i*) How do the manufacturers make their pricing decisions over time? (*ii*) How are the prices and service levels in the second period influenced by those in the first period? (*iii*) How does the whole supply chain behave over time? What indication(s) is there for us to learn about the firms' interaction and system's temporal (inter-temporal) behaviors?

Existing studies on multiple-period models can be separated into two groups. Studies in the first group are mostly from the industrial engineering and operations research community; they focus on production and/or inventory management by a single firm. The second group is mostly from the marketing and management community; they concentrate on competition and interactions among firms through either price or nonprice factor(s) over time. See Section 2 for more literature review on this topic. This research studies the inter-temporal behavior of the manufacturers and retailer in the supply chain depicted in Figure 1. Each period can be viewed as one selling season or a span over one product generation. Thus each period in our model can span over one quarter, 6 months, or 2 years, depending on the nature of the product being considered. The "inter-temporal" factors include (1) the difference in retail prices of two manufacturers' products from the previous period, and (2) the difference in the service levels provided by the two manufacturers in the previous period. We would like to investigate the impact of these factors to supply-chain partners' management decisions elaborated below.

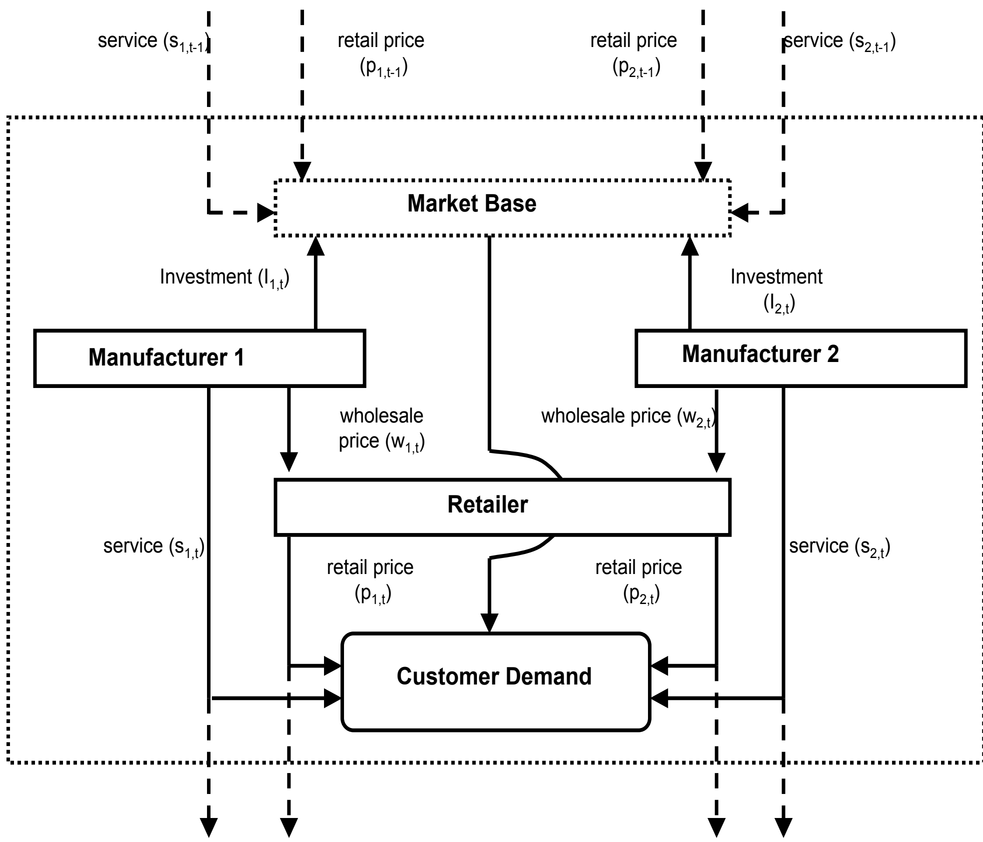

Figure 1: Supply Chain System.

This article is interested in the behavior of each firm (one retailer and two competing manufacturers) over time when faced with "learning" demand. Namely, we assume that demand for each product in any given period is affected by two types of components: (1) the difference in prices and services between the two products in the previous period, and (2) the amount of investment by each manufacturer between each period to expand the market base of its product (or brand). This assumption on the behavior of consumers demand reflects the fact that consumers have learned from the experience they had with the service provided by each manufacturer and the price they paid for the product. They also are influenced by the investment by each manufacturer to expand its product's market base (i.e., promotions, advertising campaigns, etc.).

Within each period a manufacturer has to make decisions on wholesale price, service level and amount of investment to expand its market base for that period. The decision on the amount of investment is taken at the very beginning of each period. The decisions on the wholesale price and service level are taken by each manufacturer after the market has been influenced by the investment. Finally, the retailer makes its decision on the retail price of both products at the end of each period. The decision cycle is repeated over time in this order. See Figure 2 for the timeline of events within each transaction. Note that we concentrate on the Manufacturer Stackelberg model in this research. See Charoensiriwath and Lu (2004(a)) for a study of comparing this model against Retailer Stackelberg and Nash models in a single period situation.

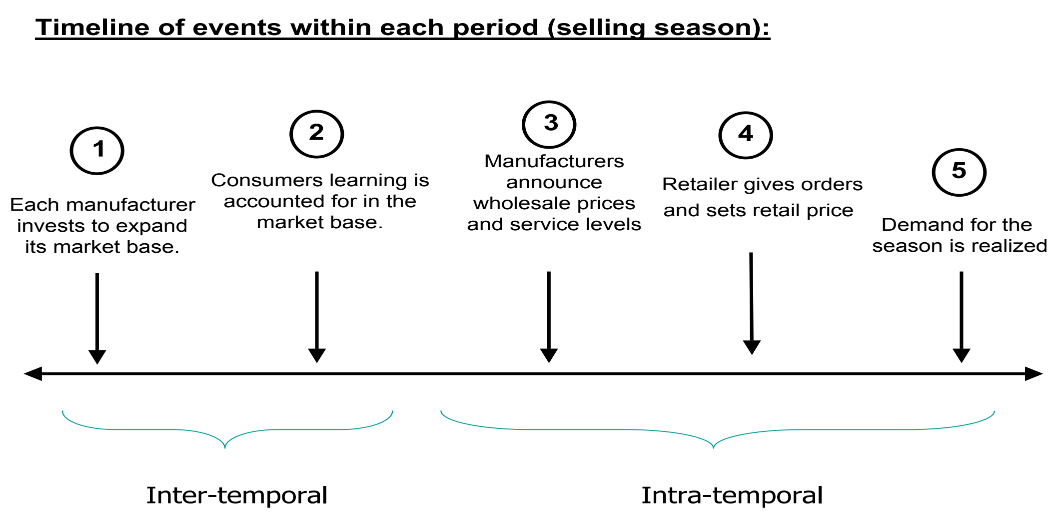

Figure 2: Timeline of events within each transaction.

Section ?? presents the notation amd model for supporting the research on impact of learning from repeated transactions. The analysis of the model using game theory and dynamic system concept is presented in Section 4 and 5, respectively. Numerical examples are given in the Section 6. Comparison of results from our model and a myopic model is presented in Section 7. Finally, Section 8 gives a few final remarks.

2 Literature Reviews on Multiple-period Models

2.1 Productions and Supply Chain Management

Early literature in this area include Thomas (1970) [21], who considered the joint pricing-production decision in a discrete-time (multi-period) setting. Federgruen and Heching (1999) [9] studied pricing-production models with concave revenue functions. Specifically, they examined a single-item problem in which a firm faces uncertain price-dependent demand. The paper addresses the simultaneous determination of pricing and inventory replenishment strategies for such a firm in both finite and infinite horizon models, with the objective of maximizing total expected discounted profit or its time average value.

There is a rich collection of literature on supply chain coordination with stochastic demand. Several mechanisms have been identified to coordinate manufacturer-retailer channels. They include the inventory buyback/return policy (Pasternack (1985) [16]), the quantity flexibility policy (Tsay (1999) [23]) for models without pricing decisions, revenue sharing contracts (Cachon and Lariviere (2000) [3]), and a two-part tariff (Weng (1997) [26]) for pricing and production decisions. The coordination mechanisms serve as means to share risk among firms in a channel in order to resolve incentive incompatibilities due to uncertainties. For an up-to-date comprehensive review on this line of research, the reader is referred to Cachon (2001) [4].

2.2 Multi-period Dynamic Competition

The majority of studies in this area are from the marketing and management community. Marketing literature models demand as diffusion of acceptance with adoption rate/sales rate and focuses on consumer adoption process of a new product. The research on diffusion models was originated with the Bass model (Bass (1969) [1]). Robinson and Lakhani (1975) [19] were the first to incorporate the variable of price into the Bass model. In recent work, the cost experience curve has been introduced on the production side; hence there are learning effects on both demand and cost. Most applications deal with durable goods where each adopter represents one unit of sales. In most cases, repeated sales have been ignored. However, Jeuland and Dolan (1982) [13] and Mahajan *et al.* (1983) [15]) included repeated purchases in their models. Dockner (1985) [8] generalized the Robinson-Lakhani model to a duopoly and applied a game-theoretic approach to find a Nash Equilibrium on the decision of product price. However, this group of literature focuses only on price as the main decision variable, i.e., no service level decision. It also does not consider the role of retailer in the supply chain during dynamic competition.

There is another parallel stream of research in economics and marketing that is not based on Bass' diffusion model. Demand is assumed to be derived from aggregated scanner data from retailers. Our model follows this approach which is common in microeconomics (see Tirole (2000) [22] and Shy (2000) [20]). Both price and nonprice variable(s) can be included in the model. Hotelling (1929) [12] was the first to introduce a formal model of product differentiation through price and location. Gabszewicz and Thisse (1979) [11] and Cohen and Whang (1997) [7] developed models where customers' preference for products can be strictly ordered (for example, quality - the higher, the better). Other studies such as Chintagunta (1993) [5] examined the sensitivity of equilibrium profits in advertising game in a duopolistic market. Chintagunta and Rao (1996) [6] considered pricing strategies in a dynamic duopoly. Fruchter and Kalish (1997) [10] investigated dynamic competition through advertisement between two firms.

3 Model

3.1 Notations and Supply Chain Descriptions

In the multiple-period model each decision variable has a subscript $t = 1, 2, ..., N$ for indicating which transaction period is considered. The subscript $i = 1$ and 2 denotes the manufacturer (or product) associated with the variable. Let $\Pi_{R,t}$ and $\Pi_{M_i,t}$ be the total profit for the retailer and manufacturer, respectively, $p_{i,t}$, $w_{i,t}$, $Q_{i,t}$ and $a_{i,t}$ be the retail price, wholesale price, demand and market size for product i, and $s_{i,t}$ and $I_{i,t}$ be the amount of service provided by supplier i to the consumer and the amouint of investment from manufacturer i to expand its market base at the beginning of the tth transaction.

There are two suppliers of competing products and each supplier manufactures one product. The two products are sold competitively to end consumers through one common retailer. The demand for each product in each period depends on two factors: (1) the difference in retail prices between the two competing products, and (2) the difference in level of service provided by the product's manufacturer and its competitor. Thus, within each period the demand for each product can be expressed as:

$$Q_{i,t} = a_{i,t} - (b_p + \theta_p)p_{i,t} + \theta_p p_{j,t} + (b_s + \theta_s)s_{i,t} - \theta_s s_{j,t}, \tag{1}$$

where $a_i > 0$, $b_p > 0$, $\theta_p > 0$, $b_s > 0$, $\theta_s > 0$, $i = 1, 2$, and $j = 3 - i$.

The positive constant a_i can be viewed as the size of a "market base" (Tsay and Agrawal 2000) which will be further elaborated in Equation (2). We assume that a_i is large enough so that Q_i will always be non-negative. We can think of $(b_p + \theta_p)$ as the measure of the responsiveness of each manufacturer's market demand to its own price. When the price of product i is decreased by one unit, the product will gain $b_p + \theta_p$ more customers. Amongst these customers, θ_p of them are switching from the competitor's product while b_p of them are the direct result of a larger market demand due to the lower price. In other words, b_p of them would not buy the product otherwise. A similar explanation can be used for service-related parameters b_s and θ_s.

Our multi-period model also takes into account the inter-temporal influence of retail prices and services on consumer demand in the next period. This is a result of the "learning" behavior of consumers. This "learning" behavior is reflected in the increase or decrease in the size of each product's market base over time (indicated by $a_{i,t}$ in the Equation (1) above). Speficically, each product's market size is affected by two inter-temporal factors: (1) the difference in retail price from the previous period, and (2) the difference in level of service provided by the manufacturers in the previous period.

In addition to these two inter-temporal factors, the manufacturers can influence the size of their product's market base by making some investment to expand its market base (*i.e.*, through advertising campaigns, improved business infrastructure, alliance formation, promotions, etc.) at the beginning of each period.

This study focuses on the manufacturer stackelberg model. Both manufacturers are Stackelberg leaders of the supply chain. That is, we assume that the suppliers in oligopolistic markets are able to establish a supplier-driven channel. In each transaction the manufacturers simultaneously announce the values of their decision variables (wholesale price and the service level) before any transaction occurs. After that the retailer reacts to the announcement by deciding what the retail price of each product should be.

Figure 2 shows the timeline of events within each period. For keeping our studies brief we assume that each manufacturer has complete information about its competitor and the retailer's cost parameters and also the consumer's demand responsiveness to the retail price. Therefore, considering the problem from Step 3 to Step 5, for given wholesale prices chosen in Step 3, the manufacturer knows the retailer's response in Step 4 and, hence, their own profit in Step 5. Each manufacturer will take this into account so as to choose the wholesale price and service level to maximize his own profit. Similar reasoning also applies when we consider the problem faced by the manufacturers in Step 1. The manufacturers can anticipate the market reaction (through size of the market base) in Step 2 when making their decisions on the amount of investment in Step 1. Furthermore, the manufacturers in Step 1 can also take into account their own best anticipated courses of action in Step 3 and the retailer reactions in Step 4 to maximize their individual profit to be realized in Step 5.

3.2 Learning Demand Function

This next equation of market-size in the demand function (1) reflects the "learning" by consumers about the experience they had gained before making their buying decisions within this period (before Step 3-5 begins).

$$a_{i,t+1} = a_{i,t} - \gamma(p_{i,t} - p_{j,t}) + \sigma(s_{i,t} - s_{j,t}) + \beta\sqrt{I_{i,t+1}} \tag{2}$$

The following provides rationale of the model formulation. To keep our model brief, we assume that consumer's memory on history of past transactions can go back to only one period. Specifically, consumers only care about the relative differences in the retail prices and service levels between the two competing products. Information on past prices and service reputation has been made available to consumers via channels such as many websites on the Internet or TV commercials. Note that Model (2) indicates that the investment in market base by a manufacturer has a decreasing return. Thus, the manufacturers can not keep investing their money to expand their market base. Moreover, the investment by one manufacturer ($I_{i,t}$) does not directly affect the market size of the other product within the same period ($a_{j,t}$). However, through strategic movement by the two manufacturers, it is possible that an indirect effect exists. Namely, an increase in investment by manufacturer i can induce more investment by manufacturer j. Our analysis of the model considers this indirect influence through the game-theoretic framework.

3.3 Manufacturers' and Retailer's Profit Functions

As in single-period problem, Manufacturer i's profit within each period is the revenue minus production cost, service cost and investment cost (e.g., Charoensiriwath and Lu, 2004(a)):

$$\Pi_{M_i,t} = (w_{i,t} - c_i)Q_{i,t} - \frac{\eta_i s_{i,t}^2}{2} - I_{i,t} \tag{3}$$

where $i = 1, 2$ and η_i is the service cost coefficient of manufacturer i. Note that this model captures the effect from the "diminishing return of service" by a quadratic form of service cost. The retailer profit is the difference between the wholesale and the retail prices of products sold as given below:

$$\Pi_{R,t} = \sum_{i=1}^{2}(p_{i,t} - w_{i,t})Q_{i,t} \tag{4}$$

where $Q_{i,t}$ is as specified in Equation (1).

In any period, all supply-chain players (vertical-interaction partners or horizontal-competition manufacturers) must ask how the decision they makes will affect other players and the results in the future periods. Game-theoretic approach is suitable to analyze this problem. Note that the analysis here requires more than just a simple repeated game-framework over multiple periods, but a combination of game theory and dynamic systems concept. The game theory is used to analyze strategic interactions among firms in the supply chain. Equilibrium can then be derived. Dynamic system concept is employed to analyze the evolving equilibrium of the supply chain over time.

To simplify the multi-period analysis of competition between manufacturers and their interaction with retailer, we assume that both manufacturers and the retailer have a "one-period look-ahead" behavior. This means that in any period t, each firm will try to maximize the sum of profits in period t and $t+1$. Vilcassim *et al.* (1999) [25] also used this framework in their analysis on firms competing on both price and advertisement[1]. This two-period optimization assumption is different than the "myopic" assumption in which firms only care for current period profit when making their decisions. It also differs from the model in which firms try to maximize their profits over all N periods (i.e., until the end of a finite time horizon). The main difficulty in the latter framework is tractability of the closed-form solution of decision variables.

The question in our framework with such a simplification is whether the *moving* two-period solution provides a reasonable approximation to the behavior of firms in the real world. To address this question, we refer to results from an empirical study by Vilcassim *et al.* (1999) [25]. They found that the relative effect of the actions from the current-period on demand of the future-period "ranged from around 18% to 9%, while the effect to the two periods into the future was at at most around 8%." Hence, the moving two-period model can be treated as a reasonable approximation to real profit maximizing behavior of firms. Studies of the infinity horizon problems will be left to our future work.

4 Analysis of the Model – Part I: Equilibrium

The equilibrium concept used in our analysis is the "subgame-perfect equilibrium" (reference xxx). Using a game-theoretic framework, the problem is solved backwards. That is, to make a decision for period t, we begin by considering the $(t+1)$st period problem. Once the reaction functions in the $(t+1)$st period are derived, the decision problems by each firm in the tth period are then derived and analyzed. The methodology in calculating reaction functions in both periods is similar. First, the reaction function (on retail price) by the retailer must be derived. Then the equilibrium wholesale price and service level given by each manufacturer are derived. Finally, the amount of investment (to induce market size) made by each manufacturer is calculated. The only difference between these calculations in both periods is that when performing the calculations in the $(t+1)$st period, we assume that firms already have information on the value of prices and service levels in the tth period. The following sections provide details.

[1] Their study uses econometric model to estimate the demand and competitive interaction parameters. Some parameters in our study rely on their study to get an estimation on the range of value.

4.1 Second (Next) Period Analysis

We solve the problem here by first separating the problem into two phrases. The first one can be called an $inter-temporal$ subproblem. This is the subproblem where the decision variables involve some variables from the previous period. This subproblem covers the Step 1 and Step 2 defined in Figure 2. The other subproblem is an $intra-temporal$ subproblem. This is the subproblem in which all the parameters and variables are the results of decisions made within the period. This subproblem covers the Step 3 to Step 5 in Figure 2. In deriving the equilibrium results, we solve the intra-temporal subproblem first.

4.1.1 Intra-Temporal Subproblem

The $intra-temporal$ subproblem in period $t+1$ is the same as the single-period problem. This is because by the time the retailer made decision on retail prices in Step 5 and the manufacturers make their decisions on wholesale prices and service levels in Step 3, the market size parameters ($a_{i,t+1}$ for $i=1,2$) have already taken into account the investment made by the manufacturers in Step 1 of period $t+1$ and any inter-temporal effects from the previous period. Thus, the objective function of each manufacturer right before the start of Step 3 does not include the investment ($I_{i,t+1}$). Therefore, the results of studies (see Charoensiriwath and Lu (2004(a)) for an example) on the Manufacturer Stackelberg model in single-period problem can be applied to the $intra-temporal$ subproblem here. That is, we will work on the single period problem by ignoring the time index $t+1$ for retailer's reaction function. Then, manufacturers make their decisions given that they know how the retailer would react.

Retailer Reaction Function

The retailer in this game must choose retail prices p_1^* and p_2^* to maximize his equilibrium profit. That is,

$$p_i^* \in arg \max_{p_i} \Pi_R(p_i, p_j^* | w_1, w_2, s_1, s_2) \tag{5}$$

where $\Pi_R(p_i, p_j | w_1, w_2, s_1, s_2)$ denotes the profit to the retailer at this stage when he sets retail prices p_i, p_j, conditioning on the earlier decisions of w_1, w_2, s_1, s_2 by the manufacturers. The first-order condition can be shown as

$$\begin{aligned} 0 = \frac{\partial \Pi_R}{\partial p_i} \quad &= \quad a_i - 2b_p p_i + \theta_p(p_j - 2p_i) + b_s s_i - \theta_s(s_j - s_i) + w_i b_p + w_i \theta_p \\ &\quad + p_j \theta_p - w_j \theta_p, \end{aligned} \tag{6}$$

where $i \in \{1, 2\}$ and $j = 3 - i$. The second-order contitions are $\partial \Pi_R^2 / \partial p_i^2 = -2b_p - 2\theta_p$, and $\partial \Pi_R^2 / \partial p_i \partial p_j = \partial \Pi_R^2 / \partial p_j \partial p_i = 2\theta_p$. Because $b_p > 0$ and $\theta_p > 0$, the system has a negative definite

Hessian. Therefore, the p_1 and p_2 calculated above are the optimal reaction functions for the retailer as follows:

$$p_i^* = \frac{w_i}{2} + \frac{(b_p+\theta_p)a_i + \theta_p a_j}{2b_p(b_p+2\theta_p)} - \frac{\theta_s(s_j - s_i)}{2(b_p+2\theta_p)} + \frac{(b_p+\theta_p)b_s s_i + \theta_p b_s s_j}{2b_p(b_p+2\theta_p)}. \tag{7}$$

Using the results from Equation (7) and (1), the demand quantities become:

$$Q_i^* = \frac{a_i}{2} - \frac{(b_p+\theta_p)}{2}w_i + \frac{\theta_p}{2}w_j + \frac{(b_s+\theta_s)}{2}s_i - \frac{\theta_s}{2}s_j, \quad i = 1, 2, j = 3 - i. \tag{8}$$

We can see that the equilibrium quantities p_i^* and Q_i^* for each product are linear functions of the wholesale prices and service levels by the manufacturers and the market sizes (a_1 and a_2).

Manufacturers Decisions

Given retailer's reaction functions, we can derive each manufacturer's optimal wholesale price and service level by maximizing each manufacturer's profit shown in Equation (3). Recall that the two manufacturers move simultaneously. Thus, a Nash Equilibrium exists between them. That is,

$$w_i^* \in arg \max_{w_i} \Pi_{M_i}(w_i, w_j^*, s_i^*, s_j^*), \quad s_i^* \in arg \max_{s_i} \Pi_{M_i}(w_i^*, w_j^*, s_i, s_j^*), \tag{9}$$

where $\Pi_{M_i}(w_i, w_j, s_i, s_j)$ is the profit of manufacturer i at this stage (time period $t+1$). To find the optimal wholesale price, w_i, we first look at the first-order condition.

$$\begin{aligned} 0 = \frac{\partial \Pi_{M_i}}{\partial w_i} &= a_i - b_p\left[w_i + \frac{(b_p+\theta_p)a_i + \theta_p a_j}{2b_p(b_p+2\theta_p)} - \frac{\theta_s(s_j-s_i)}{2(b_p+2\theta_p)} + \frac{(b_p+\theta_p)b_s s_i + \theta_p b_s s_j}{2b_p(b_p+2\theta_p)}\right] \\ &\quad + \theta_p\left[\frac{a_j - a_i}{2(b_p+2\theta_p)} + \frac{w_j - 2w_i}{2} + \frac{(2\theta_s + b_s)(s_j - s_i)}{2(b_p+2\theta_p)}\right] \\ &\quad + b_s s_i - \theta_s(s_j - s_i) + \frac{c_i b_p}{2} + \frac{c_i \theta_p}{2}, \end{aligned}$$

$$0 = \frac{\partial \Pi_{M_i}}{\partial s_i} = (w_i - c_i)\left[-\frac{b_p\theta_s}{2(b_p+2\theta_p)} - \frac{b_p(b_p+\theta_p)b_s}{2b_p(b_p+2\theta_p)} - \frac{\theta_p(b_s+2\theta_s)}{2(b_p+2\theta_p)} + b_s + \theta_s\right] - \eta_i s_i.$$

The second-order derivatives are: $\partial\Pi_{M_i}^2/\partial w_i^2 = -b_p - \theta_p$,
$\partial\Pi_{M_i}^2/\partial w_i \partial s_i = b_s + \theta_s/2$, and $\partial\Pi_{M_i}^2/\partial s_i^2 = -\eta_i$. Because $b_p > 0$ and $\theta_p > 0$, we have a negative definite Hessian. Therefore, w_i and s_i calculated above are the optimal reaction functions for the manufacturer i. Put the time period notation back and write in the "state-space" matrix form

preparing the presentation (see Section 5) in the dynamic system models (reference xxx). The manufacturer's equilibrium wholesale price and service level can then be calculated as follows:

$$\begin{bmatrix} w_{1,t+1} \\ w_{2,t+1} \end{bmatrix} = \begin{bmatrix} \varphi_1 & \varphi_1 D_2 \\ \varphi_2 D_1 & \varphi 2 \end{bmatrix} \begin{bmatrix} a_{1,t+1} \\ a_{2,t+1} \end{bmatrix} + \begin{bmatrix} \varphi_1 n_{11} & \varphi_1 n_{12} \\ \varphi_2 n_{21} & \varphi_2 n_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{10}$$

$$\begin{bmatrix} s_{1,t+1} \\ s_{2,t+1} \end{bmatrix} = \begin{bmatrix} l_1 & l_1 D_2 \\ l_2 D_1 & l_2 \end{bmatrix} \begin{bmatrix} a_{1,t+1} \\ a_{2,t+1} \end{bmatrix} + \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{11}$$

where $i, j \in \{1, 2\}$, $j \neq i$ and

$$\begin{array}{ll} A_i = 4\eta_i(b_p + \theta_p) + (b_s + \theta_s)^2 & B_i = 2\eta_i\theta_p - \theta_s(b_s + \theta_s)\left(\frac{b_p - b_s + 2\theta_p}{b_p + 2\theta_p}\right) \\ D_i = \frac{B_i}{A_i} & \\ E_i = (b_p + \theta_p) - \frac{(b_s+\theta_s)^2}{2\eta_i} & F_i = \frac{\theta_s(b_s+\theta_s)}{2\eta_i} - \frac{\theta_p b_s(b_s+\theta_s)}{2\eta_i(b_p+2\theta_p)} \\ n_{11} = E_1 + F_1 D_2 & n_{12} = F_2 + E_2 D_2 \\ n_{21} = F_1 + E_1 D_1 & n_{22} = E_2 + F_2 D_1 \\ \varphi_i = \frac{2\eta_i A_j}{A_1 A_2 - B_1 B_2} & l_i = \varphi_i \frac{(b_s+\theta_s)}{2\eta_i} \\ m_{11} = l_1(E_1 + F_1 D_2 - \frac{1}{\varphi_1}) & m_{12} = l_1(F_2 + E_2 D_2) \\ m_{21} = l_2(F_1 + E_1 D_1) & m_{22} = l_2(E_2 + F_2 D_1 - \frac{1}{\varphi_2}). \end{array}$$

With the results shown above we can calculate for the expression of the retail price ($p_{i,t}$ for $i = 1, 2$) and demand quantity ($Q_{i,t}$ for $i = 1, 2$). That is

$$\begin{bmatrix} p_{1,t+1} \\ p_{2,t+1} \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} a_{1,t+1} \\ a_{2,t+1} \end{bmatrix} + \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{12}$$

$$\begin{bmatrix} Q_{1,t+1} \\ Q_{2,t+1} \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix} \begin{bmatrix} a_{1,t+1} \\ a_{2,t+1} \end{bmatrix} + \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{13}$$

where the definition of t_{ij}, y_{ij}, g_{ij}, and h_{ij} for $i, j \in \{1, 2\}$ are given in Appendix A.

Note that the market size ($a_{i,t+1}$ for $i = 1, 2$) in Equation (10) to (13) are the market size after seeing the "learning" effect by the consumer as shown in Equation (2). To match the state-space matrix representation given above, Equation (2) is rewritten as follows:

$$\begin{bmatrix} a_{1,t+1} \\ a_{2,t+1} \end{bmatrix} = \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} + \begin{bmatrix} -\gamma & \gamma \\ \gamma & -\gamma \end{bmatrix} \begin{bmatrix} p_{1,t} \\ p_{2,t} \end{bmatrix} + \begin{bmatrix} \sigma & -\sigma \\ -\sigma & \sigma \end{bmatrix} \begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} + \begin{bmatrix} \beta\sqrt{I_{1,t+1}} \\ \beta\sqrt{I_{2,t+1}} \end{bmatrix} \tag{14}$$

In the next section, the amount of investment ($I_{i,t+1}$) each manufacturer should invest to influence the market size will be derived.

4.1.2 Inter-Temporal Decisions

The objective function of the manufacturers at this stage is as shown in Equation (3). The manufacturer i must choose the investment $I^*_{i,t+1}$ to maximize its equilibrium profit. Denoted the vectors by $\boldsymbol{p_t} = [p_{1,t}, p_{2,t}]$, $\boldsymbol{w_t} = [w_{1,t}, w_{2,t}]$, $\boldsymbol{s_t} = [s_{1,t}, s_{2,t}]$, $\boldsymbol{I_t} = [I_{1,t}, I_{2,t}]$. The investment $I^*_{i,t+1}$ at equilibrium can be expressed as

$$I^*_{i,t+1} \in arg \max_{I_{i,t+1}} \Pi_{M_i,t+1}(I_{i,t+1}, I^*_{j,t+1} | \boldsymbol{p_t}, \boldsymbol{w_t}, \boldsymbol{s_t}, \boldsymbol{I_t}), \tag{15}$$

where $\Pi_{M_i,t+1}(I_{i,t+1}, I^*_{j,t+1} | \boldsymbol{p_t}, \boldsymbol{w_t}, \boldsymbol{s_t}, \boldsymbol{I_t})$ is the profit of manufacturer i at time period $t+1$, conditioning on earlier decisions on $\boldsymbol{p_t}$, $\boldsymbol{w_t}$, $\boldsymbol{s_t}$ and $\boldsymbol{I_t}$ from time period t. Using Equation (14), the first-order conditions can be shown as

$$\begin{aligned} 0 \;=\; & \Bigg\{ 2\eta_i Q_{i,t+1} K_j + (w_{i,t+1} - c_i) \Bigg[\frac{1}{2} - (b_p + \theta_p)\eta_i K_j + \theta_p \eta_j D_i K_i + \frac{(b_s+\theta_s)^2}{2} K_j \\ & - \frac{\theta_s}{2}(b_s + \theta_s) D_i K_i \Bigg] - \eta_i (b_s + \theta_s) K_j s_{i,t+1} \Bigg\} \frac{\beta}{2\sqrt{I_{i,t+1}}}, \end{aligned} \tag{16}$$

where $i, j \in \{1, 2\}$ and $j \neq i$.

Given Equation (16) and the results from Equation (10), (11), and (13), one can derive the following linear relationship between the square root of the investment amount and the market size, retail prices and service levels from previous period t.

$$\begin{aligned} \begin{bmatrix} \sqrt{I_{1,t+1}} \\ \sqrt{I_{2,t+1}} \end{bmatrix} = & \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} + \begin{bmatrix} (\delta_{12} - \delta_{11})\gamma & -(\delta_{12} - \delta_{11})\gamma \\ (\delta_{22} - \delta_{21})\gamma & -(\delta_{22} - \delta_{21})\gamma \end{bmatrix} \begin{bmatrix} p_{1,t} \\ p_{2,t} \end{bmatrix} \\ & + \begin{bmatrix} -(\delta_{12} - \delta_{11})\sigma & (\delta_{12} - \delta_{11})\sigma \\ -(\delta_{22} - \delta_{21})\sigma & (\delta_{22} - \delta_{21})\sigma \end{bmatrix} \begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} + \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}, \end{aligned} \tag{17}$$

where the definition of δ_{ij}, and Δ_i for $i, j \in \{1, 2\}$ are given in Appendix B.

Finally, the market size in period $t+1$ ($a_{i,t+1}$) (which is the market size after seeing the effect from the investment $I_{1,t+1}$ and $I_{2,t+1}$) can be derived by substituting Equation (17) into Equation (2). As a result, the market size in period $t+1$ can be expressed as follows:

$$\begin{aligned} \begin{bmatrix} a_{1,t+1} \\ a_{2,t+1} \end{bmatrix} = & \begin{bmatrix} (\delta_{12} - \delta_{11})\gamma\beta - \gamma & -(\delta_{12} - \delta_{11})\gamma\beta + \gamma \\ (\delta_{22} - \delta_{21})\gamma\beta + \gamma & -(\delta_{22} - \delta_{21})\gamma\beta - \gamma \end{bmatrix} \begin{bmatrix} p_{1,t} \\ p_{2,t} \end{bmatrix} \\ & + \begin{bmatrix} -(\delta_{12} - \delta_{11})\sigma\beta + \sigma & (\delta_{12} - \delta_{11})\sigma\beta - \sigma \\ -(\delta_{22} - \delta_{21})\sigma\beta - \sigma & (\delta_{22} - \delta_{21})\sigma\beta + \sigma \end{bmatrix} \begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} \\ & + \begin{bmatrix} (\beta\delta_{11} + 1) & \beta\delta_{12} \\ \beta\delta_{21} & (\beta\delta_{22} + 1) \end{bmatrix} \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} + \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}. \end{aligned} \tag{18}$$

4.2 First (Current) Period Analysis

After knowing how each firm will behave in period $t+1$ conditioning on the information on decision in period t, we next consider the decisions faced by each firm in period t.

4.2.1 Intra-Temporal Subproblem

Retailer Reaction Function

At this stage the retailer needs to find the retail price $p_{1,t}^*$ and $p_{2,t}^*$ for maximizing its profit over two time periods. That is, set $p_{i,t}^* \in arg\max_{p_{i,t}} \Pi_{R,t}(p_{i,t}, p_{j,t}^*|\boldsymbol{w_t}, \boldsymbol{s_t}, \boldsymbol{I_t})$ with

$$\Pi_{R,t}(\boldsymbol{p_t}|\boldsymbol{w_t}, \boldsymbol{s_t}, \boldsymbol{I_t}) \quad = \quad \sum_{\tau=t}^{t+1}\sum_{i=1}^{2}(p_{i,\tau} - w_{i,\tau})Q_{i,\tau} \tag{19}$$

For this we obtain the following first-order derivatives:

$$\begin{aligned}
\frac{\partial \Pi_{R,t}}{\partial p_{i,t}} \quad = \quad & Q_{i,t} + (p_{i,t} - w_{i,t})\frac{\partial Q_{i,t}}{\partial p_{i,t}} + (p_{j,t} - w_{j,t})\frac{\partial Q_{j,t}}{\partial p_{i,t}} \\
& + \left(\frac{\partial p_{i,t+1}}{\partial p_{i,t}} - \frac{\partial w_{i,t+1}}{\partial p_{i,t}}\right)Q_{i,t+1} + (p_{i,t+1} - w_{i,t+1})\frac{\partial Q_{i,t+1}}{\partial p_{i,t}} \\
& + \left(\frac{\partial p_{j,t+1}}{\partial p_{i,t}} - \frac{\partial w_{j,t+1}}{\partial p_{i,t}}\right)Q_{j,t+1} + (p_{j,t+1} - w_{j,t+1})\frac{\partial Q_{j,t+1}}{\partial p_{i,t}}.
\end{aligned}$$

Equate the above to zeros and check the second-order conditions. The retailer's reaction function to wholesale prices and service levels in period t can be derived as

$$\begin{bmatrix} p_{1,t} \\ p_{2,t} \end{bmatrix} \quad = \quad \begin{bmatrix} \psi_{11} & \psi_{12} \\ \psi_{21} & \psi_{22} \end{bmatrix}\begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} + \begin{bmatrix} \zeta_{11} & \zeta_{12} \\ \zeta_{21} & \zeta_{22} \end{bmatrix}\begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} + \begin{bmatrix} \Upsilon_1 \\ \Upsilon_2 \end{bmatrix}, \tag{20}$$

where the expressions for ψ_{ij}, ζ_{ij}, and Υ_i for $i, j \in \{1, 2\}$ and $j \neq i$ are given in Appendix C.

Manufacturers Decision Process

The retailer's reaction function in Equation (20) gives the manufacturers information on how their decisions will affect the retail prices and their profits. The manufacturers then use this information to set wholesale prices and service levels to maximize their individual profits over two periods (t and $t+1$). Each manufacturer's objective at this stage can be expressed as

$$\Pi_{M_i,t} \quad = \quad \sum_{\tau=t}^{t+1}\left[(w_{i,\tau} - c_i)Q_{i,\tau} - \frac{\eta_i s_{i,\tau}^2}{2} - I_{i,\tau}\right], \tag{21}$$

for $i, j \in \{1, 2\}$ and $j \neq i$. Note that in Equation (21), $I_{i,t}$ is a constant. This is because when the manufacturers make their decisions on the wholesale prices and service levels, the decisions on $I_{1,t}$

and $I_{2,t}$ have already been made. Each manufacturer must choose the wholesale price and service level to maximize its own objective. That is,

$$w_{i,t}^* \in arg \max_{w_{i,t}} \Pi_{M_i,t}(w_{i,t}, w_{j,t}^*, s_{i,t}^*, s_{j,t}^* | \boldsymbol{I_t}), s_{i,t}^* \in arg \max_{s_{i,t}} \Pi_{M_i,t}(w_{i,t}^*, w_{j,t}^*, s_{i,t}, s_{j,t}^* | \boldsymbol{I_t}). \tag{22}$$

The first-order conditions for each $i = 1, 2$ can be derived as follows:

$$\begin{aligned} 0 &= \frac{\partial \Pi_{M_i,t}}{\partial w_{i,t}} = Q_{i,t} + (w_{i,t} - c_i)\frac{\partial Q_{i,t}}{\partial w_{i,t}} + \frac{\partial w_{i,t+1}}{\partial w_{i,t}} Q_{i,t+1} \\ &\quad + (w_{i,t+1} - c_i)\frac{\partial Q_{i,t+1}}{\partial w_{i,t}} - \eta_i s_{i,t+1}\frac{\partial s_{i,t+1}}{\partial w_{i,t}} - \frac{\partial I_{i,t+1}}{\partial w_{i,t}}, \\ 0 &= \frac{\partial \Pi_{M_i,t}}{\partial s_{i,t}} = (w_{i,t} - c_i)\frac{\partial Q_{i,t}}{\partial s_{i,t}} - \eta_i s_{i,t} + \frac{\partial w_{i,t+1}}{\partial s_{i,t}} Q_{i,t+1} \\ &\quad + (w_{i,t+1} - c_i)\frac{\partial Q_{i,t+1}}{\partial s_{i,t}} - \eta_i s_{i,t+1}\frac{\partial s_{i,t+1}}{\partial s_{i,t}} - \frac{\partial I_{i,t+1}}{\partial s_{i,t}}. \end{aligned}$$

Solving the above equations and check the second-order conditions, the expression for $w_{i,t}$ and $s_{i,t}$ can be derived as linear functions of $a_{i,t}$ and c_i (for $i \in \{1, 2\}$).

$$\begin{bmatrix} w_{1,t} \\ w_{2,t} \end{bmatrix} = \begin{bmatrix} \kappa_{11} & \kappa_{12} \\ \kappa_{21} & \kappa_{22} \end{bmatrix} \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} + \begin{bmatrix} \nu_{11} & \nu_{12} \\ \nu_{21} & \nu_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{23}$$

$$\begin{bmatrix} s_{1,t} \\ s_{2,t} \end{bmatrix} = \begin{bmatrix} \vartheta_{11} & \vartheta_{12} \\ \vartheta_{21} & \vartheta_{22} \end{bmatrix} \begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} + \begin{bmatrix} \varsigma_{11} & \varsigma_{12} \\ \varsigma_{21} & \varsigma_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{24}$$

The expressions for $\vartheta_{ij}, \varsigma_{ij}$, κ_{ij}, and ν_{ij} for $i \in \{1, 2\}$ and $j = 3 - i$ are given in Appendix D.

4.2.2 Inter-Temporal Subproblem

Continuing working backwards, the next step is for the manufacturers to analyze their decisions on the level of investment $I_{i,t}$. Manufacturer i's objective to maximize for its equilibrium profit at this stage can be expressed as:

$$\Pi_{M_i,t} = \sum_{\tau=t}^{t+1} \delta^{\tau-t}\Big[(w_{i,\tau} - c_i)Q_{i,\tau} - \frac{\eta_i s_{i,\tau}^2}{2} - I_{i,\tau}\Big], \tag{25}$$

where $i \in \{1, 2\}$ and δ is a “discount factor”. For simplicity, we assume that δ is one from now on. Here $I_{i,t}$ ($i \in \{1, 2\}$) are decision variables. The first-order condition from Equation (25) with respect to the investment $I_{i,t}$ is then derived as follows:

$$\begin{aligned} 0 &= (w_{i,t} - c_i)\frac{\partial Q_{i,t}}{\partial I_{i,t}} + \frac{\partial w_{i,t}}{\partial I_{i,t}} Q_{i,t} - \eta_i s_{i,t}\frac{\partial s_{i,t}}{\partial I_{i,t}} + (w_{i,t+1} - c_i)\frac{\partial Q_{i,t+1}}{\partial I_{i,t}} \\ &\quad + \frac{\partial w_{i,t+1}}{\partial I_{i,t}} Q_{i,t+1} - \eta_i s_{i,t+1}\frac{\partial s_{i,t+1}}{\partial I_{i,t}} - \frac{\partial I_{i,t+1}}{\partial I_{i,t}} - 1. \end{aligned} \tag{26}$$

Solving Equation (26) and check the second-order conditions, the investment $I^*_{i,t}$ must satisfy the following relationship:

$$\begin{bmatrix} \sqrt{I^*_{1,t}} \\ \sqrt{I^*_{2,t}} \end{bmatrix} = \begin{bmatrix} \varpi_{11} & \varpi_{12} \\ \varpi_{21} & \varpi_{22} \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix} + \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{21} & \varepsilon_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{27}$$

where the expressions for ϖ_{ij}, and ε_{ij} for $i, j \in \{1, 2\}$ and $j \neq i$ are given in Appendix D.

Using the fact that the market size in the t^{th} period is

$$a_{i,t} = a_{i,t-1} - \gamma(p_{i,t-1} - p_{j,t-1}) + \sigma(s_{i,t-1} - s_{j,t-1}) + \beta\sqrt{I_{i,t}}. \tag{28}$$

We can derive the system equation for the market size as:

$$\begin{bmatrix} a_{1,t} \\ a_{2,t} \end{bmatrix} = \begin{bmatrix} \chi_{11} & \chi_{12} \\ \chi_{21} & \chi_{22} \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix} + \begin{bmatrix} \omega_{11} & \omega_{12} \\ \omega_{21} & \omega_{22} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{29}$$

where the expressions for χ_{ij}, and ω_{ij} for $i, j \in \{1, 2\}$ and $j \neq i$ are given in Appendix D. The result in Equation (29) is based on the assumption that $a_{i,0} = a_i$ where a_i is the initial market size given for product i, and $p_{i,0} = s_{i,0} = 0$ for $i \in \{1, 2\}$.

5 Analysis of The Model – Part II: Dynamic Systems

Equation (29) governs the dynamics of market sizes and production cost over time. Alternatively, we can write it in the following form:

$$\begin{bmatrix} a_{1,t} \\ a_{2,t} \\ c_{1,t} \\ c_{2,t} \end{bmatrix} = \begin{bmatrix} \chi_{11} & \chi_{12} & \omega_{11} & \omega_{12} \\ \chi_{21} & \chi_{22} & \omega_{21} & \omega_{22} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \\ c_{1,t-1} \\ c_{2,t-1} \end{bmatrix}. \tag{30}$$

or in matrix notations,

$$\mathbf{\Phi(t) = M\Phi(t-1)}. \tag{31}$$

Equation (31) represents a homogeneous dynamic system. Note that the production cost $c_{1,t}$ and $c_{2,t}$ does not change over time. From the system equation above, a "*system equilibrium point*" can be defined in the following definition. See reference xxx for details.

DEFINITION 5.1. *A vector $\overline{\mathbf{\Phi}}$ is an equilibrium point of a dynamic system if it has the property that once the system state vector is equal to $\overline{\mathbf{\Phi}}$ it remains equal to $\overline{\mathbf{\Phi}}$ for all future time.*

That is, with the assumption that $\mathbf{\Phi}(\mathbf{0}) \neq \mathbf{0}$, a system equilibrium point must satisfy the condition

$$\overline{\mathbf{\Phi}} = \mathbf{M}\overline{\mathbf{\Phi}}. \tag{32}$$

Equation (32) is useful to find the system equilibrium when one exists. From the dynamic system theory, the existence of system equilibrium depends on the value of the dominant eigenvalue of $\mathbf{M}$. Specifically, the following lemma follows directly from the dynamic system theory (see Luenberger (1979) [14], page number xxx for details) and the structure of homogeneous dynamic system of market sizes stated in Equation (30) and (31). The dominant eigenvalue of $\mathbf{M}$ is the eigenvalue with the largest absolute value. Subdominant eigenvalues of $\mathbf{M}$ refer to all other eigenvalues of $\mathbf{M}$.

LEMMA 5.1. *(Luenberger [1979]) Long-term behavior of the market sizes is determined by the dominant eigenvalue of* $\boldsymbol{M}$*. Subdominant eigenvalues of* $\boldsymbol{M}$ *determine how quickly the market sizes converge or diverge.*

From Lemma 5.1, we can analyze the dynamic behavior of the whole supply chain (*i.e.*, retail prices, wholesale prices and service levels) through the dynamic behavior of market sizes governed by Equation (31). Section 4 has express all decision variables in a period as a function of market sizes and production costs in that period. The results are given as follows:

THEOREM 5.1. *Let* λ_D *be the dominant eigenvalue of* $\boldsymbol{M}$*, then* λ_D *equals*

(a) 1, if $|\chi_{11}|$ *and* $|\chi_{22}|$ *are less than one;*

(b) χ_{11}*, if* $|\chi_{11}|$ *is great than one and* $|\chi_{22}|$*;*

(c) χ_{22}*, if* $|\chi_{22}|$ *is greater than one and* $|\chi_{11}|$*,*

where χ_{11} *and* χ_{22} *are as defined in Equation (30).*

Proof: Eigenvalues of $\mathbf{M}$ are scalars λ such that $\mathbf{M}-\lambda\mathbf{I}$ is singular. This implies one must solve for λ that satisfies $det(\mathbf{M}-\lambda\mathbf{I}) = 0$. However, from the definition of $\mathbf{M}$ in Equation (30), eigenvalues of $\mathbf{M}$ are $1, \chi_{11}$, and χ_{22}. Thus, the dominant eigenvalue must be the biggest of these three numbers. □

The following theorem states the condition that governs the convergent or divergent behavior of the system.

THEOREM 5.2. *If the dominant eigenvalue of* $\boldsymbol{M}$ *is less than one, the market sizess of both products converge to a constant over time. If the dominant eigenvalue is greater one, the market sizes diverge. If the dominant eigenvalue equals to 1, the system can converge or diverge.*

Proof: See Appendix E.

Theorem 5.2 states the dynamic behavior of the market sizes of both products over time. Although the theorem states only the behavior of market sizes, other quantities such as wholesale prices, retail prices, service levels, and demand quantity also follow the behavior of market sizes. If the market bases converge, these variables will converge as well. Likewise, if the market sizes diverge, they will also diverge.

Even when we know exactly whether the market sizes will converge or diverge, the dynamic behavior of the market sizes (and prices and service levels) between periods can vary. For example, there can be some oscillation in the market sizes before each of them converge to a value. Alternatively, the market size can smoothly increase or decrease to a value over time. In the first case, the leader-follower roles can be alternating between the two products. Namely, the two products can switch the market leader-follower role[2] during the oscillation and before they reach the final convergent value. The following theorem states the conditions that govern the period-by-period behavior of the market sizes.

THEOREM 5.3. *The dynamic behavior of market sizes is goverened by the value of its dominant eigenvalue as follows:*

(i) If every eigenvalue of $\boldsymbol{M}$ is positive, the dynamic pattern of market sizes is a geometric sequence of the form r^k, which (increasingly) diverge if $r > 1$ and converge if $r \leq 1$. No oscillation occurs in this case.

(ii) If there is at least one eigenvalue that is negative, the response is an alternating geometric sequence of the form $(-1)^k|r|^k$. If $|r| < 1$, market sizes will converge with decreasing oscillations. If $|r| > 1$, the market sizes diverge with increasing oscillations.

Proof: See Appendix F.

Theorem 5.3 characterizes the period-by-period behavior of the system variables. When the dominant eigenvalue is negative, the market leader-follower roles between the two manufacturers can be alternating *every period* due to the oscillation in the system variables (i.e., market sizes, prices, service levels). When the dominant eigenvalue is real and positive, it is still possible that the two manufacturers switch their market leadership. However, this switching can *occur only once* since there will be no oscillation in the system variables. Figure 3 shows the situation when all eigenvalues are positive. The dominant eigenvalue of $\mathbf{M}$ is 1.0000, while the subdominant eigenvalue equals 0.059. The system smoothly converges to the system equilibrium predicted by Equation (32).

Figure 4 shows the situation when the dominant eigenvalue is positive and greater than 1 but the subdominant eigenvalue is negative and greater than -1. In this example, the dominant eigenvalue of $\mathbf{M}$ is 1.0046, while the subdominant eigenvalue equals -0.5671. As seen from the

[2]A firm holds market leadership if it has larger market base than its competition.

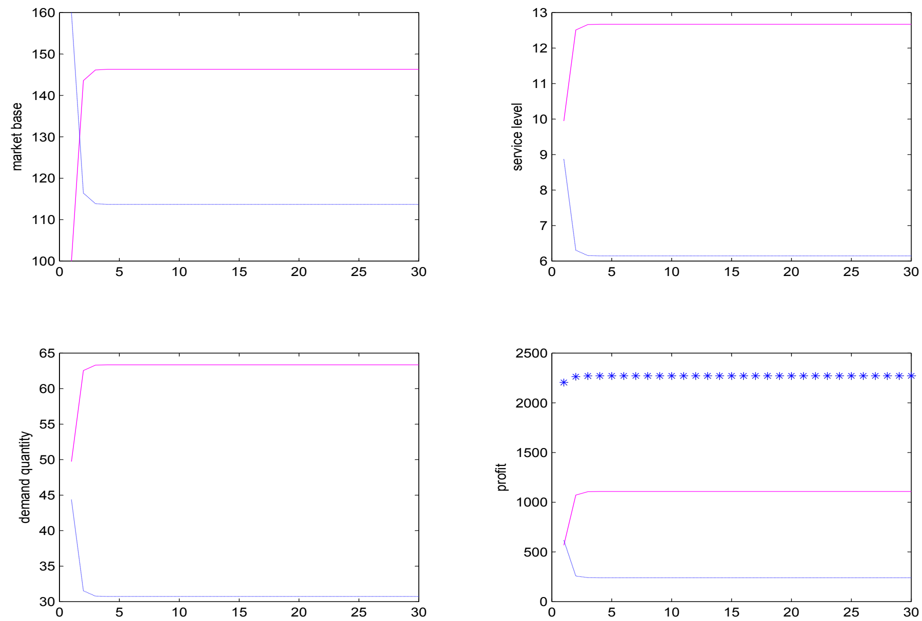

Figure 3: The evolution of equilibrium market bases when $\beta = 0.0, \gamma = 2.8, \sigma = 2.8$, $b_p = b_s = 2, \theta_p = \theta_s = 1.1, a_{1,0} = 100, a_{2,0} = 160, c_1 = 5, c_2 = 15, \eta_1 = \eta_2 = 5$ (Manufacturer 1: Red, Manufacturer 2: Blue, Retailer: *).

figure, the system oscillates during the first few periods before it settles on a smoother increasing behavior. Note that Manufacturer 1 starts off being a market leader but ends up by being a market follower. Detailed discussions on this behavior of the two manufacturers will be presented in the next section.

6 Numerical Studies

This section shows numerical examples of a few important real-life cases for providing observations and managerial insights. The range of the model parameters are taken from the existing literature (e.g., Tsay and Agrawal (2000) [24] and Vilcassin *et al.* (1999) [25]). See Appendix G for details.

6.1 No Oscillation

Based on the results given in Equation (31) and Theorem 5.3, occurrence of oscillation behavior of market size depends on model parameter values. Our numerical studies will explore a range of parameter values such that the system smoothly moves to the equilibrium as defined in Equation

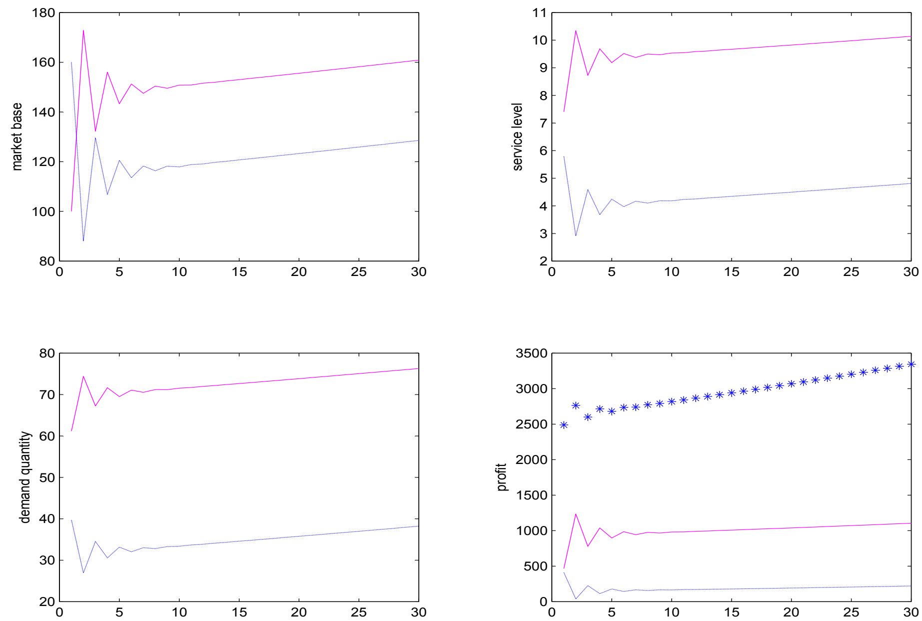

Figure 4: The evolution of equilibrium market bases when $\beta = 0.3, \gamma = 3.8, \sigma = 2.8$, $b_p = b_s = 2, \theta_p = \theta_s = 1.1, a_{1,0} = 100, a_{2,0} = 160, c_1 = 5, c_2 = 15, \eta_1 = \eta_2 = 5$ (Manufacturer 1: Red, Manufacturer 2: Blue, Retailer: *).

(32). The following observation gives the conditions such that oscillation in market behavior would not occur.

OBSERVATION 6.1. *The oscillation behavior of the supply chain system will not occur if all the following conditions hold:*

(a) $\gamma \leq \max(b_p, \theta_p)$, *(b)* $\sigma \leq \max(b_s, \theta_s)$ *and (c)* $0.5\sigma \leq \gamma \leq 1.5\sigma$.

This observation states that if values of both γ and σ are not very different from each other and not far from b_p, θ_p, b_s, and θ_s, market size's evolution over time will be smooth without oscillations. Part (a) and (b) are reasonable and valid in most situations since demand should be more sensitive to current price (service) than last period price (service). To understand part (c), we should examine the case when this condition is not satisfied. If demand is much more sensitive to last period price than last period service ($\gamma >> \sigma$), a two-period profit-maximizing manufacturer may sell product at a low price in period t and plan to overprice in period $t+1$. However, when period $t+1$ is reached, the manufacturer will find that it has a smaller market base in period $t+1$ due to overpricing. It then would have to underprice again in period $t+2$ in order to regain the market base loss due to overpricing in period $t+1$. This phenomenon would repeat itself overtime and cause oscillation in market bases, prices, and service levels. A similar situation can occur when demand is more sensitive to last period service than the last period price. Thus, in a situation where demand sensitivities to prices and service levels are not far from each other, there will be no oscillation in the system.

Now, consider the situations given in Figure 4. The only difference in parameter values between these situations is the β value. When $\beta = 0$, the investment $I_{i,t}$ will not affect the market size for product i in period t for $i = 1, 2$. Namely,

$$a_{i,t} = a_{i,t-1} - \gamma(p_{i,t-1} - p_{j,t-1}) + \sigma(s_{i,t-1} - s_{j,t-1}), \quad (33)$$

for $i, j \in \{1, 2\}$ and $j \neq i$. This is the case when consumers are not sensitive to the investment made by the manufacturers in the current period. Thus, it is not beneficial to the manufacturers to invest in any market expansion activities ($I_{i,t} = 0$ for $i \in 1, 2$). On the other hand, if $\beta > 0$, the consumers are sensitive to the market investment. Thus, it is always beneficial for the manufacturers to invest some money for market expansion in this case. The following observation captures both scenarios.

OBSERVATION 6.2. *The β value of consumer sensitivity to market expansion investment determines whether the system is convergent or divergent:*

- *(i) When $\beta = 0$ there will be no investment to expand market bases in any period and the system will converge.*
- *(ii) When $\beta > 0$ the manufacturers will keep investing in expanding the market bases and the system will diverge.*

Figure 3 shows the situation when $\beta = 0$. It shows that the system finally becomes stable. When $\beta > 0$, both manufacturers will keep investing to expand their market bases. In that case, the market base will keep growing as shown in Figure 4.

To the end of this article, we assume the validity of conditions given in Observation 6.1 on the range of γ and σ. This is to prevent oscillations in market sizes over time.

6.2 Service-Emphasized Market

Figure 5 shows a typical case where demand is more sensitive to last period service than last period price in the learning process[3]. In this case, we find that the firm with service cost advantage will be the winner over time. This result assures the importance of service component in competitions over repeated transactions.

[3] It is unrealistic to consider market with service only sensitivity and ignore the price component all together (i.e., $\beta = \gamma = 0$). At least that is not the product type we are concentrating on with our model.

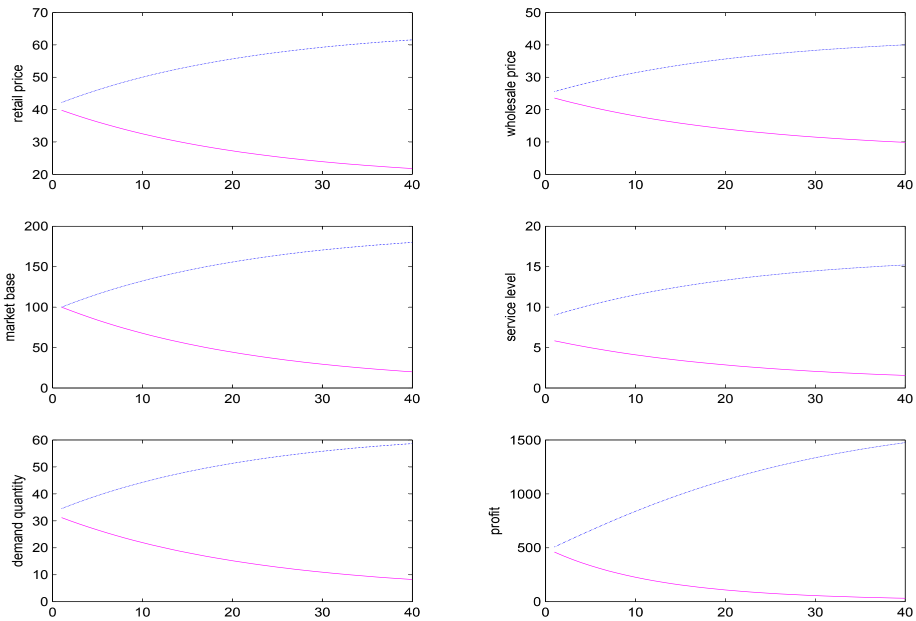

Figure 5: The evolution of equilibrium market bases when $\beta = 0.0, \gamma = 0.6, \sigma = 1.8$, $b_p = b_s = 2, \theta_p = \theta_s = 1.1, a_1 = 100, a_2 = 100, c_1 = 5$, $c_2 = 5, \eta_1 = 7$, $\eta_2 = 5$ (Manufacturer 1: Red, Manufacturer 2: Blue).

6.3 Price Sensitive Market

This example investigates the result of a special case where consumers only care about prices from previous period in their learning process. Namely, $\beta = \sigma = 0$ while $\gamma > 0$. Since $\beta = 0$, the system will finally converge to a system equilibrium. In this case, the final retail price of both products will be the same. However, the company with the cost advantage (either production or service cost) can afford to sell its product cheaper while providing more service to consumers. The retailer will sell both products at the same price. This leads to an equilibrium in which the company with the cost advantage gets more demand for its product and earns greater profit. This situation emphasizes the role of the retailer as a middle man who can control the consumer demand through retail price setting. Figure 6 shows the system dynamics in a typical price-sensitive market. The following observation states this result.

OBSERVATION 6.3. *Given that demand is only sensitive to price in its learning process (i.e., $\beta = \sigma = 0$ and $\gamma > 0$), the company with any type of cost advantage will gain more profit and capture a larger market base than its competitor. The retailer will sell both products at the same retail price but the firm with cost advantage will be able to support more service to its customers.*

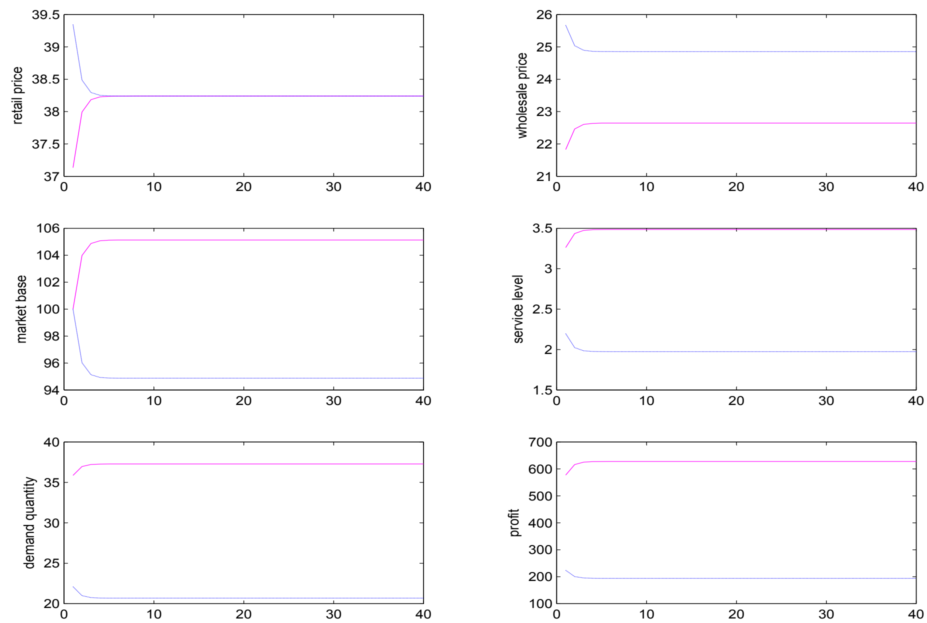

Figure 6: The evolution of equilibrium market bases when $\beta = 0.0, \gamma = 1.8, \sigma = 0.0$, $b_p = b_s = 2, \theta_p = \theta_s = 1.1, a_1 = 100, a_2 = 100, c_1 = 5,\ c_2 = 15, \eta_1 = 5,\ \eta_2 = 5$ (Manufacturer 1: Red, Manufacturer 2: Blue).

6.4 Identical Manufacturers

This example considers the situation when the two manufacturers are identical in product and service cost. Namely, $c_1 = c_2 = c$ and $\eta_1 = \eta_2 = \eta$. We observe that no matter how different the initial value the manufacturers have for market sizes, both products will be sold at the same price with the same level of service provided to the consumers. An example of this scenario is shown in Figure 7. The observation stated below emphasizes the importance of production and service cost in competition between the two manufacturers over repeated transactions. If the two manufacturers possess similar underlying production and service capability, initial advantage by either company on the market base vanishes over time.

OBSERVATION 6.4. *If all the costs are the same, the two manufacturers will converge to the same market size and sell their products at the same price, while providing equal level of service to consumers. This happens even though the two products may start with different market sizes initially.*

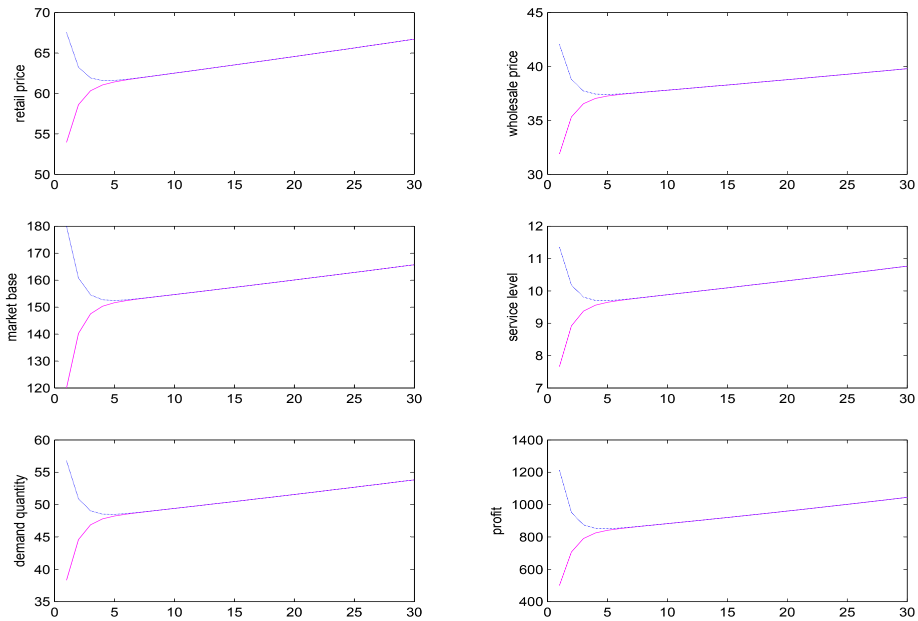

Figure 7: The evolution of equilibrium market bases when $\beta = 0.3, \gamma = 2, \sigma = 2$, $b_p = b_s = 2, \theta_p = \theta_s = 1.1, a_1 = 120, a_2 = 80, c_1 = c_2 = 15, \eta_1 = \eta_2 = 5$ (Manufacturer 1: Red, Manufacturer 2: Blue, Retailer: *).

6.5 Production Cost Leader Versus Service Cost Leader

This example studies the competition between two manufacturers that possess different advantages. One manufacturer, company 1, possesses superior production technology and thus has a lower production cost. The other, company 2, is more efficient in providing service. Thus, $c_1 < c_2$ and $\eta_1 > \eta_2$. We are interested in investigating the extent to which each advantage can help a company to compete with the other.

The following observation states that the company with service cost advantage will always win in the long-run over the company with production cost advantage, no matter how large the production cost advantage or how small the service cost advantage. This observation emphasizes the importance of service component in competition over the long-run. Figure 8 shows a typical situation in the competition between a production cost leader and a service cost leader. Notice that at the beginning, the service cost leader may have a smaller demand and earn less profit. However, as it keeps increasing service levels to consumers, it can finally win more customers and earn bigger profit than its production cost leader competitor. Note also that the production cost leader company has a larger initial market base but that still does not change the end result.

OBSERVATION 6.5. *Given that demand is equally sensitive to both price and service level (i.e., $b_p = b_s$, $\theta_p = \theta_s$, $\gamma = \sigma > 0$) and $\beta > 0$, the company with service cost advantage may earn less profit and capture smaller market base in the beginning. However, it will finally gain more profit and capture larger market base than its competitor with a smaller production cost. This happens no matter how large the production cost advantage company 1 has over company 2, or how small the service cost advantage company 2 has over company 1.*

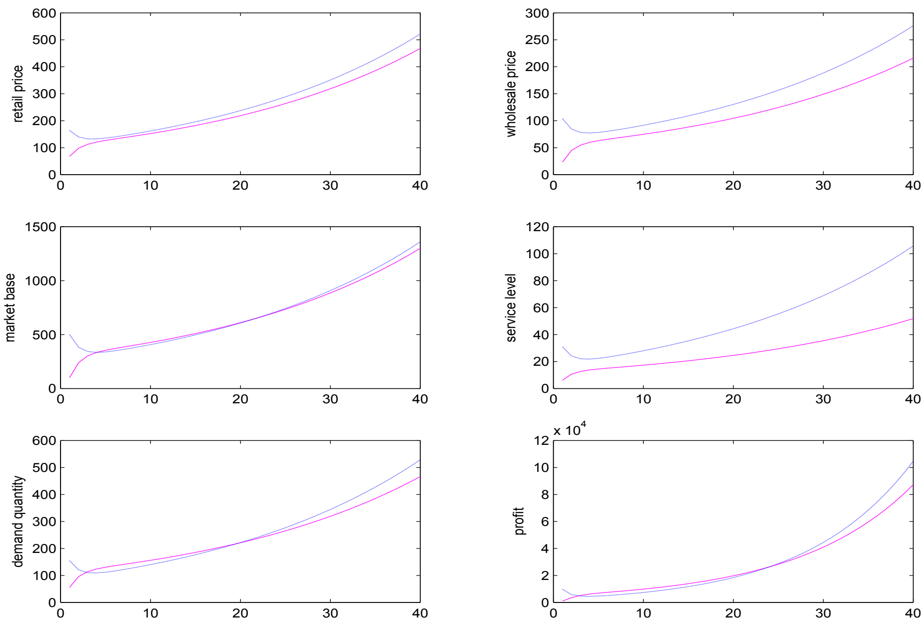

Figure 8: The evolution of equilibrium market bases when $\beta = 0.9, \gamma = 1.8, \sigma = 1.8$, $b_p = b_s = 2, \theta_p = \theta_s = 1.1, a_1 = 100, a_2 = 500, c_1 = 3$, $c_2 = 25, \eta_1 = 9$, $\eta_2 = 5$ (Manufacturer 1: Red, Manufacturer 2: Blue).

7 Comparisons with the Myopic Model

This section compares our "two-period learning" models with myopic models. In the myopic model, the two manufacturers and the retailer just try to optimize their single-period profits. For making the comparison sensible by not giving advantge to the leaning model, the comparison is conducted under two different assumptions on demand. In the first case, demand is memoryless. Consumers in this case do not learn from past experience and only concern about prices and services in current period. For our model, this is a special case when $\beta = \gamma = \sigma = 0$. In the second case, consumers learn from past experience. Thus, demand for product i will depend on prices and service in both the previous and the current periods. To begin our comparison, we first state the equilibrium decisions made by myopic firms.

Myopic Decision Model

Myopic firms optimize profit in the current period. In comparison to our model, it is as if firms are in the second stage of the two-period profit-optimizing model studied in Section 4.1. Therefore, the results from Section 4.1 can be applied here. Thus, for any time period t, manufacturers' investment to expand market base within current period can be calculated by

$$\begin{bmatrix} \sqrt{I_{1,t}} \\ \sqrt{I_{2,t}} \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{12} \\ \delta_{21} & \delta_{22} \end{bmatrix} \begin{bmatrix} a_{1,t-1} \\ a_{2,t-1} \end{bmatrix} + \begin{bmatrix} (\delta_{12}-\delta_{11})\gamma & -(\delta_{12}-\delta_{11})\gamma \\ (\delta_{22}-\delta_{21})\gamma & -(\delta_{22}-\delta_{21})\gamma \end{bmatrix} \begin{bmatrix} p_{1,t-1} \\ p_{2,t-1} \end{bmatrix}$$
$$+ \begin{bmatrix} -(\delta_{12}-\delta_{11})\sigma & (\delta_{12}-\delta_{11})\sigma \\ -(\delta_{22}-\delta_{21})\sigma & (\delta_{22}-\delta_{21})\sigma \end{bmatrix} \begin{bmatrix} s_{1,t-1} \\ s_{2,t-1} \end{bmatrix} + \begin{bmatrix} \Delta_1 \\ \Delta_2 \end{bmatrix}.$$

Moreover, from Section 4.1 the wholesale prices, retail prices, service levels, and demand quantities for both products can be calculated. All the parameters are as defined previously in Section 4.1. We now compare the numerical results from our model and the myopic model.

7.1 Myopic Firms with Memoryless Demand

Figure 9 shows the comparison in this case. The market sizes are the same for both models and do not change over time since there is no learning demand. The manufacturers do not have to invest since demand is not affected by their investments ($\beta = 0$). It can be seen that the manufacturers' profits are higher in our model. Service levels and prices are also higher in our model, even though demand is smaller. Thus, the manufacturers in our model concentrate on the higher end of the market (high service or high price), whereas the manufacturers in myopic model focus on the lower end (low service or low price). This is an important insight for firms in a market where the learning effect from consumers is small. High-end consumers are willing to pay more for higher service level and firms can earn more profits focusing on this group of consumers.

7.2 Myopic Firms with Learning Demand

Figure 10 shows the comparison when consumers learn from the past period. In a myopic model, the firms only care about their profits in the current period and ignore any future effects their behavior might cause over time. Thus, they are not capable to cope with the learning consumers. Their markets shrink and they earn less profit over time. On the other hand, our model, with think-ahead firms, can prevent this phenomenon from happening. They plan their actions to take advantage of the learning behavior of demand. The service levels and prices are chosen such that the firms are rewarded by the consumers. Thus, markets keep growing for both products while

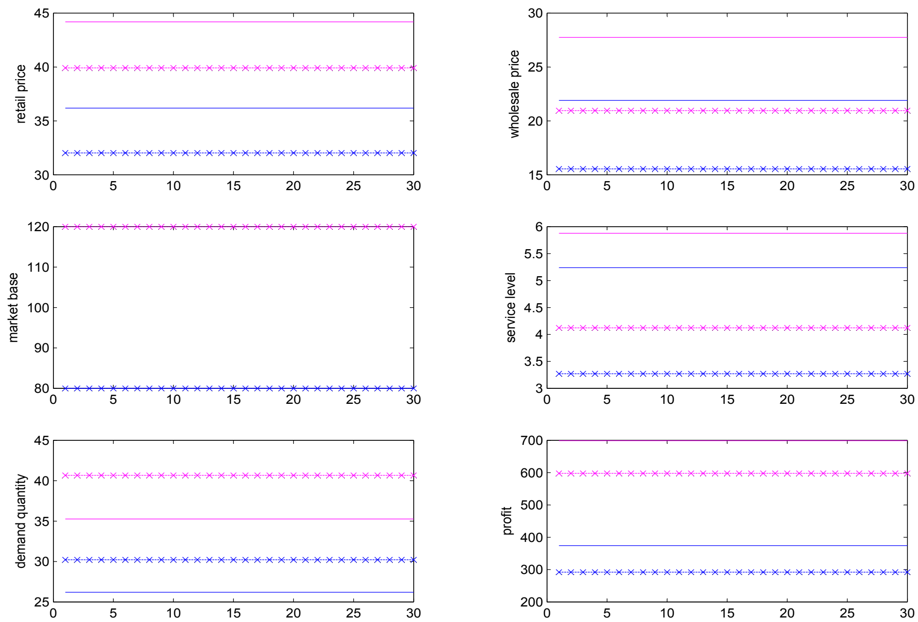

Figure 9: Comparison between Myopic and Two-period profit optimizing model. $\beta = 0, \gamma = 0, \sigma = 0$, $b_p = b_s = 2, \theta_p = \theta_s = 1.1, a_1 = 120, a_2 = 80, c_1 = 5, c_2 = 5, \eta_1 = 6, \eta_2 = 5$ (Myopic: xxx, Two-Period: –, Manufacturer 1: Red, Manufacturer 2: Blue).

firms can keep earning more profits.

8 Conclusion

Both game theory and dynamic system concept are used to characterize our model of learning from repeated transactions. We assume that firms use a moving two-period profit-maximizing strategy. Demand is assumed to have a "learning" capability. Information on the previous period prices and services, as well as manufacturers' investment to expand market bases, can influence market size of each product in the current period. Using concepts from dynamic systems with numerical studies on several interesting cases, several important managerial insights are obtained.

We find that if demand is only sensitive to price in the learning process and all the costs are the same between two identical manufacturers, the manufacturers will possess equal market size even though they may initially start with different market sizes. Our main finding is that if demand is equally sensitive to both price and service level, the manufacturer with service cost advantage may earn less profit and capture a smaller market base in the beginning. However, it will finally gain

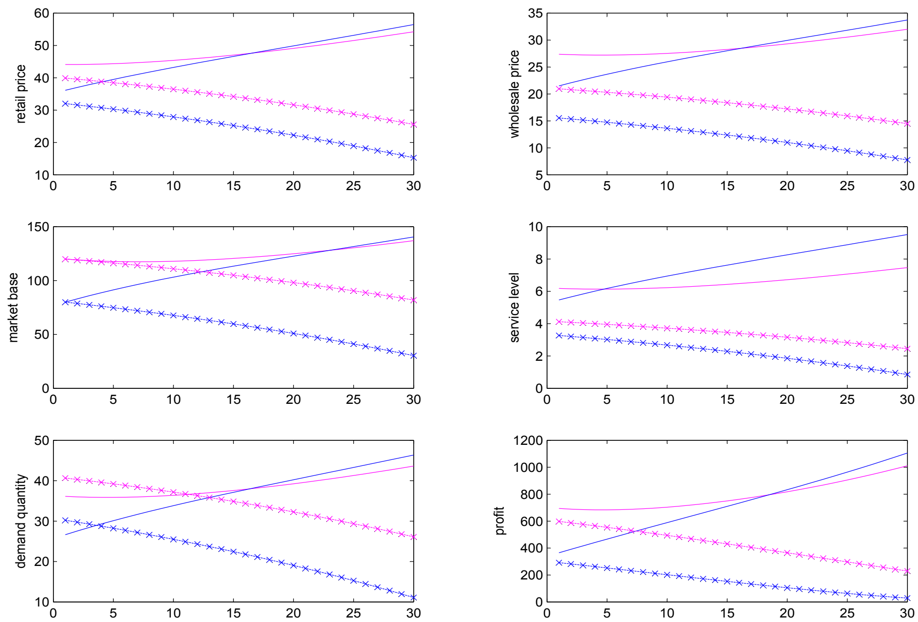

Figure 10: Comparison between Myopic and Two-period profit optimizing model. $\beta = 0.5, \gamma = 0.4, \sigma = 0.3$, $b_p = b_s = 2, \theta_p = \theta_s = 1.1, a_1 = 120, a_2 = 80, c_1 = 5, c_2 = 5, \eta_1 = 6, \eta_2 = 5$ (Myopic: xxx, Two-Period: –, Manufacturer 1: Red, Manufacturer 2: Blue).

more profit and capture a larger market base than its competitor with a smaller production cost. This happens no matter how large the production cost advantage its competitor has, or how small the service cost advantage the manufacturer has over its competitor.

We realize that our assumption on constant unit production cost over time may not be realistic. Other alternatives such as economy-of-scale production cost or decreasing return-to-scale production cost can be explored in the future. These assumptions will affect the pricing behavior of both products over time. In our case, since unit production cost is constant, a firm can increase service levels and keep charging a higher price without worrying much about production cost. Thus, retail price can keep increasing as long as service can make up for the price increase. Other assumptions on production cost are likely to yield different results.

Appendix A. Parameters Specifications for Equation (12) and (13)

$$G = \frac{\theta_s b_p + b_s(b_p+\theta_p)}{2b_p(b_p+2\theta_p)}, \qquad H = \frac{\theta_p b_s - \theta_s b_p}{2b_p(b_p+2\theta_p)}$$

$$t_{11} = \frac{\varphi_1}{2} + \frac{(b_p+\theta_p)}{2b_p(b_p+2\theta_p)} + Gl_1 + Hl_2D_1, \qquad t_{12} = \frac{\varphi_1 D_2}{2} + \frac{\theta_p}{2b_p(b_p+2\theta_p)} + Gl_1D_2 + Hl_2$$

$$t_{21} = \frac{\varphi_2 D_1}{2} + \frac{\theta_p}{2b_p(b_p+2\theta_p)} + Hl_1 + Gl_2D_1, \qquad t_{22} = \frac{\varphi_2}{2} + \frac{(b_p+\theta_p)}{2b_p(b_p+2\theta_p)} + Hl_1D_2 + Gl_2$$

$$y_{11} = \varphi_1\frac{E_1+F_1D_2}{2} + Gm_{11} + Hm_{21}, \qquad y_{12} = \varphi_1\frac{F_2+E_2D_2}{2} + Gm_{12} + Hm_{22}$$

$$y_{21} = \varphi_2\frac{F_1+E_1D_1}{2} + Hm_{11} + Gm_{21}, \qquad y_{22} = \varphi_2\frac{E_2+F_2D_1}{2} + Hm_{12} + Gm_{22}$$

$$g_{11} = \frac{1}{2} - \frac{b_p+\theta_p}{2}\varphi_1 + \frac{\theta_p}{2}\varphi_2D_1 + \frac{b_s+\theta_s}{2}l_1 - \frac{\theta_s}{2}l_2D_1,$$

$$g_{12} = -\frac{b_p+\theta_p}{2}\varphi_1D_2 + \frac{\theta_p}{2}\varphi_2 + \frac{b_s+\theta_s}{2}l_1D_2 - \frac{\theta_s}{2}l_2,$$

$$g_{21} = -\frac{b_p+\theta_p}{2}\varphi_2D_1 + \frac{\theta_p}{2}\varphi_1 + \frac{b_s+\theta_s}{2}l_2D_1 - \frac{\theta_s}{2}l_1,$$

$$g_{22} = \frac{1}{2} - \frac{b_p+\theta_p}{2}\varphi_2 + \frac{\theta_p}{2}\varphi_1D_2 + \frac{b_s+\theta_s}{2}l_2 - \frac{\theta_s}{2}l_1D_2,$$

$$h_{11} = -\frac{b_p+\theta_p}{2}\varphi_1(E_1+F_1D_2) + \frac{\theta_p}{2}\varphi_2(F_1+E_1D_1) + \frac{b_s+\theta_s}{2}m_{11} - \frac{\theta_s}{2}m_{21}$$

$$h_{12} = -\frac{b_p+\theta_p}{2}\varphi_1(F_2+E_2D_2) + \frac{\theta_p}{2}\varphi_2(E_2+F_2D_1) + \frac{b_s+\theta_s}{2}m_{12} - \frac{\theta_s}{2}m_{22}$$

$$h_{21} = -\frac{b_p+\theta_p}{2}\varphi_2(F_1+E_1D_1) + \frac{\theta_p}{2}\varphi_1(E_1+F_1D_2) + \frac{b_s+\theta_s}{2}m_{21} - \frac{\theta_s}{2}m_{11}$$

$$h_{22} = -\frac{b_p+\theta_p}{2}\varphi_2(E_2+F_2D_1) + \frac{\theta_p}{2}\varphi_1(F_2+E_2D_2) + \frac{b_s+\theta_s}{2}m_{22} - \frac{\theta_s}{2}m_{12}$$

Appendix B. Parameters Specifications for Equation (17)

$$\alpha_{i,t} = a_{i,t} - \gamma(p_{i,t} - p_{j,t}) + \sigma(s_{i,t} - s_{j,t})$$
$$\phi_i = (E_i + F_i D_j)c_i + (F_j + E_j D_j)c_j$$
$$\mu_i = \frac{A_i(b_s+\theta_s)}{A_1A_2 - B_1B_2}$$
$$\lambda_i = \left[\frac{A_i(b_s+\theta_s)(F_i+E_iD_i)}{A_1A_2-B_1B_2}\right]c_i + \left[\frac{A_i(E_j+F_jD_i)}{A_1A_2-B_1B_2} - \frac{1}{2\eta_j}\right](b_s+\theta_s)c_j$$
$$\tau_i = \frac{1}{2}\left[(b_s+\theta_s)\mu_j - (b_p+\theta_p)\varphi_i + \theta_p\varphi_j D_i - \theta_s\mu_i D_i\right]$$
$$\upsilon_i = \frac{1}{2}\left[(b_s+\theta_s)\mu_j D_j - (b_p+\theta_p)\varphi_i D_j + \theta_p\varphi_j - \theta_s\mu_i\right]$$
$$\varrho_i = -\frac{(b_p+\theta_p)\varphi_i}{2}\phi_i + \frac{\theta_p\varphi_j}{2}\phi_j - \frac{\theta_s}{2}\lambda_i + \frac{(b_s+\theta_s)}{2}\lambda_j$$
$$\rho_i = (\tfrac{1}{2} + \tau_i)\alpha_{i,t} + \upsilon_i\alpha_{j,t} + \varrho_i$$
$$\Lambda_i = \varphi_i\rho_i + \xi_i\varphi_i(\alpha_{i,t} + Dj\alpha_{j,t} + \phi_i) - \eta_i\mu_j^2(\alpha_{i,t} + Dj\alpha_{j,t}) - \eta_i\mu_j\lambda_j - \xi_1 c_1$$
$$r_i = \varphi_i(0.5 + 2\tau_i) - \eta_i\mu_j^2$$
$$\Psi_i = \varphi_i\upsilon_i + \varphi_i(0.5+\tau_i)D_j - \eta_i\mu_j^2 D_j$$
$$\delta_{11} = \frac{\beta(2-\beta^2 r_2)(0.5\varphi_1+r_1)+\beta^3\Psi_1\Psi_2}{(2-\beta^2r_1)(2-\beta^2r_2)-\beta^4\Psi_1\Psi_2}, \quad \delta_{12} = \frac{\beta(2-\beta^2r_2)\Psi_1+\beta^3\Psi_1(0.5\varphi_2+r_2)}{(2-\beta^2r_1)(2-\beta^2r_2)-\beta^4\Psi_1\Psi_2}$$
$$\delta_{21} = \frac{\beta(2-\beta^2 r_1)\Psi_2+\beta^3\Psi_2(0.5\varphi_1+r_1)}{(2-\beta^2r_1)(2-\beta^2r_2)-\beta^4\Psi_1\Psi_2}, \quad \delta_{22} = \frac{\beta(2-\beta^2r_1)(0.5\varphi_2+r_2)+\beta^3\Psi_1\Psi_2}{(2-\beta^2r_1)(2-\beta^2r_2)-\beta^4\Psi_1\Psi_2}$$
$$\Delta_1 = \frac{\beta(2-\beta^2r_2)\Omega_1+\beta^3\Psi_1\Omega_2}{(2-\beta^2r_1)(2-\beta^2r_2)-\beta^4\Psi_1\Psi_2}, \quad \Omega_1 = \varphi_1\varrho_1 + (0.5+\tau_1)\varphi_1\phi_1 - \eta_1\mu_2\lambda_2 - (0.5+\tau_1)c_1$$
$$\Delta_2 = \frac{\beta^3 r_2\Omega_1+\beta(2-\beta^2r_1)\Omega_2}{(2-\beta^2r_1)(2-\beta^2r_2)-\beta^4\Psi_1\Psi_2}, \quad \Omega_2 = \varphi_2\varrho_2 + (0.5+\tau_2)\varphi_2\phi_2 - \eta_2\mu_1\lambda_1 - (0.5+\tau_2)c_2$$
$$\widehat{\nu} = \widehat{\nu_1}^2 - \widehat{\nu_2}^2$$
$$\widehat{\alpha_1} = \beta(\delta_{12} - \delta_{11} - \tfrac{1}{\beta}), \quad \widehat{\alpha_2} = \beta(\delta_{22} - \delta_{21} + \tfrac{1}{\beta})$$

Appendix C. Parameters Specifications for Equation (20)

$$\psi_{11} = \widehat{\nu}(\widehat{\nu_2}\widehat{\vartheta_3} - \widehat{\nu_3}\widehat{\vartheta_2}), \qquad \psi_{12} = \widehat{\nu}(\widehat{\nu_2}\widehat{\vartheta_4} - \widehat{\nu_4}\widehat{\vartheta_2})$$

$$\psi_{21} = \widehat{\nu}(\widehat{\nu_1}\widehat{\vartheta_3} - \widehat{\nu_3}\widehat{\vartheta_1}), \qquad \psi_{22} = \widehat{\nu}(\widehat{\nu_1}\widehat{\vartheta_4} - \widehat{\nu_4}\widehat{\vartheta_1})$$

$$\zeta_{11} = -\widehat{\nu}(\theta_p\widehat{\nu_2} + (b_p + \theta_p)\widehat{\vartheta_2}), \qquad \zeta_{12} = \widehat{\nu}((b_p + \theta_p)\widehat{\nu_2} + \theta_p\widehat{\vartheta_2})$$

$$\zeta_{21} = \widehat{\nu}(\theta_p\widehat{\nu_1} + (b_p + \theta_p)\widehat{\vartheta_1}), \qquad \zeta_{22} = -\widehat{\nu}((b_p + \theta_p)\widehat{\nu_1} + \theta_p\widehat{\vartheta_1})$$

$$\Upsilon_1 = \widehat{\nu}(\widehat{\nu_2}\widehat{\vartheta_5} + \widehat{\nu_5}\widehat{\vartheta_2}), \qquad \Upsilon_2 = \widehat{\nu}(\widehat{\nu_5}\widehat{\vartheta_1} + \widehat{\nu_1}\widehat{\vartheta_5})$$

$$\widehat{\kappa_1} = t_{11}\gamma\widehat{\alpha_1} + t_{12}\gamma\widehat{\alpha_2}, \qquad \widehat{\kappa_2} = \varphi_1\gamma\widehat{\alpha_1} + \varphi_1 D_2\gamma\widehat{\alpha_2}$$

$$\widehat{\kappa_3} = g_{11}\gamma\widehat{\alpha_1} + g_{12}\gamma\widehat{\alpha_2}, \qquad \widehat{\kappa_4} = t_{21}\gamma\widehat{\alpha_1} + t_{22}\gamma\widehat{\alpha_2}$$

$$\widehat{\kappa_5} = \varphi_2 D_1\gamma\widehat{\alpha_1} + \varphi_2\gamma\widehat{\alpha_2}, \qquad \widehat{\kappa_6} = g_{21}\gamma\widehat{\alpha_1} + g_{22}\gamma\widehat{\alpha_2}$$

$$\widehat{\mu_1} = -t_{11}\sigma\widehat{\alpha_1} - t_{12}\sigma\widehat{\alpha_2}, \qquad \widehat{\mu_2} = -\varphi_1\sigma\widehat{\alpha_1} - \varphi_1 D_2\sigma\widehat{\alpha_2}$$

$$\widehat{\mu_3} = -g_{11}\sigma\widehat{\alpha_1} - g_{12}\sigma\widehat{\alpha_2}, \qquad \widehat{\mu_4} = -t_{21}\sigma\widehat{\alpha_1} - t_{22}\sigma\widehat{\alpha_2}$$

$$\widehat{\mu_5} = -\varphi_2 D_1\sigma\widehat{\alpha_1} - \varphi_2\sigma\widehat{\alpha_2}, \qquad \widehat{\mu_6} = -g_{21}\sigma\widehat{\alpha_1} - g_{22}\sigma\widehat{\alpha_2}$$

$$\widehat{\phi_1} = (\beta\delta_{11} + 1)a_{1,t-1} + \beta\delta_{12}a_{2,t-1} + \beta\Delta_1, \qquad \widehat{\phi_2} = (\beta\delta_{22} + 1)a_{2,t-1} + \beta\delta_{21}a_{1,t-1} + \beta\Delta_1$$

$$\widehat{\psi_1} = g_{11}\widehat{\phi_1} + g_{12}\widehat{\phi_2} + j_{11}c_1 + j_{12}c_2, \qquad \widehat{\psi_2} = g_{21}\widehat{\phi_1} + g_{22}\widehat{\phi_2} + j_{21}c_1 + j_{22}c_2$$

$$\widehat{\nu_1} = -2(b_p + \theta_p) + 2\widehat{\kappa_3}(\widehat{\kappa_1} - \widehat{\kappa_2}) + 2\widehat{\kappa_6}(\widehat{\kappa_4} - \widehat{\kappa_5}), \qquad \widehat{\nu_2} = 2\theta_p - 2\widehat{\kappa_3}(\widehat{\kappa_1} - \widehat{\kappa_2}) - 2\widehat{\kappa_6}(\widehat{\kappa_4} - \widehat{\kappa_5})$$

$$\widehat{\nu_3} = (b_s + \theta_s) + 2\widehat{\mu_3}(\widehat{\kappa_1} - \widehat{\kappa_2}) + 2\widehat{\mu_6}(\widehat{\kappa_4} - \widehat{\kappa_5}), \qquad \widehat{\nu_4} = -\theta_s - 2\widehat{\mu_3}(\widehat{\kappa_1} - \widehat{\kappa_2}) - 2\widehat{\mu_6}(\widehat{\kappa_4} - \widehat{\kappa_5})$$

$$\widehat{\nu_5} = a_{1,t} + \widehat{\psi_1}(\widehat{\kappa_1} - \widehat{\kappa_2}) + \widehat{\kappa_3}\widehat{\tau_1} + \widehat{\psi_2}(\widehat{\kappa_4} - \widehat{\kappa_5}) + \widehat{\kappa_6}\widehat{\tau_2}$$

$$\widehat{\vartheta_1} = 2\theta_p - 2\widehat{\kappa_3}(\widehat{\kappa_1} - \widehat{\kappa_2}) - 2\widehat{\kappa_6}(\widehat{\kappa_4} - \widehat{\kappa_5}),$$

$$\widehat{\vartheta_2} = -2(b_p + \theta_p) + 2\widehat{\kappa_3}(\widehat{\kappa_1} - \widehat{\kappa_2}) + 2\widehat{\kappa_6}(\widehat{\kappa_4} - \widehat{\kappa_5}),$$

$$\widehat{\vartheta_3} = -\theta_s - 2\widehat{\mu_3}(\widehat{\kappa_1} - \widehat{\kappa_2}) - 2\widehat{\mu_6}(\widehat{\kappa_4} - \widehat{\kappa_5}),$$

$$\widehat{\vartheta_4} = (b_s + \theta_s) + 2\widehat{\mu_3}(\widehat{\kappa_1} - \widehat{\kappa_2}) + 2\widehat{\mu_6}(\widehat{\kappa_4} - \widehat{\kappa_5}),$$

$$\widehat{\vartheta_5} = a_{2,t} - \widehat{\psi_1}(\widehat{\kappa_1} - \widehat{\kappa_2}) - \widehat{\kappa_3}\widehat{\tau_1} - \widehat{\psi_2}(\widehat{\kappa_4} - \widehat{\kappa_5}) - \widehat{\kappa_6}\widehat{\tau_2}$$

Appendix D. Parameters Specifications for Equation (23) and (24)

$$\kappa_{11} = \frac{\widetilde{\nu_{51}}\widetilde{\rho_4} - \widetilde{\phi_{51}}\widetilde{\rho_1}}{\widetilde{\rho_1}\widetilde{\rho_2} - \widetilde{\rho_3}\widetilde{\rho_4}}, \qquad \kappa_{12} = \frac{\widetilde{\nu_{52}}\widetilde{\rho_4} - \widetilde{\phi_{52}}\widetilde{\rho_1}}{\widetilde{\rho_1}\widetilde{\rho_2} - \widetilde{\rho_3}\widetilde{\rho_4}}$$

$$\kappa_{21} = \frac{\widetilde{\nu_{51}}\widetilde{\rho_2} - \widetilde{\phi_{51}}\widetilde{\rho_3}}{\widetilde{\rho_3}\widetilde{\rho_4} - \widetilde{\rho_1}\widetilde{\rho_2}}, \qquad \kappa_{22} = \frac{\widetilde{\nu_{52}}\widetilde{\rho_2} - \widetilde{\phi_{52}}\widetilde{\rho_3}}{\widetilde{\rho_3}\widetilde{\rho_4} - \widetilde{\rho_1}\widetilde{\rho_2}}$$

$$\nu_{11} = \frac{\widetilde{\nu_{53}}\widetilde{\rho_4} - \widetilde{\phi_{53}}\widetilde{\rho_1}}{\widetilde{\rho_1}\widetilde{\rho_2} - \widetilde{\rho_3}\widetilde{\rho_4}}, \qquad \nu_{12} = \frac{\widetilde{\nu_{54}}\widetilde{\rho_4} - \widetilde{\phi_{54}}\widetilde{\rho_1}}{\widetilde{\rho_1}\widetilde{\rho_2} - \widetilde{\rho_3}\widetilde{\rho_4}}$$

$$\nu_{21} = \frac{\widetilde{\nu_{53}}\widetilde{\rho_2} - \widetilde{\phi_{53}}\widetilde{\rho_3}}{\widetilde{\rho_3}\widetilde{\rho_4} - \widetilde{\rho_1}\widetilde{\rho_2}}, \qquad \nu_{22} = \frac{\widetilde{\nu_{54}}\widetilde{\rho_2} - \widetilde{\phi_{54}}\widetilde{\rho_3}}{\widetilde{\rho_3}\widetilde{\rho_4} - \widetilde{\rho_1}\widetilde{\rho_2}}$$

$$\vartheta_{11} = \frac{\lambda_{51}\widetilde{\rho_8} - \theta_{51}\widetilde{\rho_5}}{\widetilde{\rho_5}\widetilde{\rho_6} - \widetilde{\rho_7}\widetilde{\rho_8}}, \qquad \vartheta_{12} = \frac{\lambda_{52}\widetilde{\rho_8} - \theta_{52}\widetilde{\rho_5}}{\widetilde{\rho_5}\widetilde{\rho_6} - \widetilde{\rho_7}\widetilde{\rho_8}}$$

$$\vartheta_{21} = \frac{\xi_{51}\widetilde{\rho_{12}} - \widetilde{\varphi_{51}}\widetilde{\rho_9}}{\widetilde{\rho_9}\widetilde{\rho_{10}} - \widetilde{\rho_{11}}\widetilde{\rho_{12}}}, \qquad \vartheta_{22} = \frac{\xi_{52}\widetilde{\rho_{12}} - \widetilde{\varphi_{52}}\widetilde{\rho_9}}{\widetilde{\rho_9}\widetilde{\rho_{10}} - \widetilde{\rho_{11}}\widetilde{\rho_{12}}}$$

$$\varsigma_{11} = \frac{\lambda_{53}\widetilde{\rho_8} - \theta_{53}\widetilde{\rho_5}}{\widetilde{\rho_5}\widetilde{\rho_6} - \widetilde{\rho_7}\widetilde{\rho_8}}, \qquad \varsigma_{12} = \frac{\lambda_{54}\widetilde{\rho_8} - \theta_{52}\widetilde{\rho_5}}{\widetilde{\rho_5}\widetilde{\rho_6} - \widetilde{\rho_7}\widetilde{\rho_8}}$$

$$\varsigma_{21} = \frac{\xi_{53}\widetilde{\rho_{12}} - \widetilde{\varphi_{53}}\widetilde{\rho_9}}{\widetilde{\rho_9}\widetilde{\rho_{10}} - \widetilde{\rho_{11}}\widetilde{\rho_{12}}}, \qquad \varsigma_{22} = \frac{\xi_{54}\widetilde{\rho_{12}} - \widetilde{\varphi_{54}}\widetilde{\rho_9}}{\widetilde{\rho_9}\widetilde{\rho_{10}} - \widetilde{\rho_{11}}\widetilde{\rho_{12}}}$$

where,

$$\widetilde{\nu_{51}} = (\widetilde{\delta_{51}}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_{51}})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2}) - (\widetilde{\delta_2}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_2})(\widetilde{\beta_{51}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{51}})$$
$$\widetilde{\nu_{52}} = (\widetilde{\delta_{52}}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_{52}})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2}) - (\widetilde{\delta_2}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_2})(\widetilde{\beta_{52}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{52}})$$
$$\widetilde{\nu_{53}} = (\widetilde{\delta_{53}}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_{53}})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2}) - (\widetilde{\delta_2}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_2})(\widetilde{\beta_{53}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{53}})$$
$$\widetilde{\nu_{54}} = (\widetilde{\delta_{54}}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_{54}})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2}) - (\widetilde{\delta_2}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_2})(\widetilde{\beta_{54}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{54}})$$

$$\widetilde{\phi_{51}} = (\widetilde{\gamma_{51}}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_{51}})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2}) - (\widetilde{\gamma_2}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_2})(\widetilde{\beta_{51}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{51}})$$
$$\widetilde{\phi_{52}} = (\widetilde{\gamma_{52}}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_{52}})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2}) - (\widetilde{\gamma_2}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_2})(\widetilde{\beta_{52}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{52}})$$
$$\widetilde{\phi_{53}} = (\widetilde{\gamma_{53}}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_{53}})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2}) - (\widetilde{\gamma_2}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_2})(\widetilde{\beta_{53}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{53}})$$
$$\widetilde{\phi_{54}} = (\widetilde{\gamma_{54}}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_{54}})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2}) - (\widetilde{\gamma_2}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_2})(\widetilde{\beta_{54}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{54}})$$

$$\widetilde{\xi_{51}} = (\widetilde{\delta_{51}}\widetilde{\alpha_4} - \widetilde{\delta_4}\widetilde{\alpha_{51}})(\widetilde{\beta_2}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_2}) - (\widetilde{\delta_2}\widetilde{\alpha_4} - \widetilde{\delta_4}\widetilde{\alpha_2})(\widetilde{\beta_{51}}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_{51}})$$
$$\widetilde{\xi_{52}} = (\widetilde{\delta_{52}}\widetilde{\alpha_4} - \widetilde{\delta_4}\widetilde{\alpha_{52}})(\widetilde{\beta_2}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_2}) - (\widetilde{\delta_2}\widetilde{\alpha_4} - \widetilde{\delta_4}\widetilde{\alpha_2})(\widetilde{\beta_{52}}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_{52}})$$
$$\widetilde{\xi_{53}} = (\widetilde{\delta_{53}}\widetilde{\alpha_4} - \widetilde{\delta_4}\widetilde{\alpha_{53}})(\widetilde{\beta_2}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_2}) - (\widetilde{\delta_2}\widetilde{\alpha_4} - \widetilde{\delta_4}\widetilde{\alpha_2})(\widetilde{\beta_{53}}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_{53}})$$
$$\widetilde{\xi_{54}} = (\widetilde{\delta_{54}}\widetilde{\alpha_4} - \widetilde{\delta_4}\widetilde{\alpha_{54}})(\widetilde{\beta_2}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_2}) - (\widetilde{\delta_2}\widetilde{\alpha_4} - \widetilde{\delta_4}\widetilde{\alpha_2})(\widetilde{\beta_{54}}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_{54}})$$

$$\widetilde{\varphi_{51}} = (\widetilde{\gamma_{51}}\widetilde{\alpha_4} - \widetilde{\gamma_4}\widetilde{\alpha_{51}})(\widetilde{\beta_2}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_2}) - (\widetilde{\gamma_2}\widetilde{\alpha_4} - \widetilde{\gamma_4}\widetilde{\alpha_2})(\widetilde{\beta_{51}}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_{51}})$$
$$\widetilde{\varphi_{52}} = (\widetilde{\gamma_{52}}\widetilde{\alpha_4} - \widetilde{\gamma_4}\widetilde{\alpha_{52}})(\widetilde{\beta_2}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_2}) - (\widetilde{\gamma_2}\widetilde{\alpha_4} - \widetilde{\gamma_4}\widetilde{\alpha_2})(\widetilde{\beta_{52}}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_{52}})$$
$$\widetilde{\varphi_{53}} = (\widetilde{\gamma_{53}}\widetilde{\alpha_4} - \widetilde{\gamma_4}\widetilde{\alpha_{53}})(\widetilde{\beta_2}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_2}) - (\widetilde{\gamma_2}\widetilde{\alpha_4} - \widetilde{\gamma_4}\widetilde{\alpha_2})(\widetilde{\beta_{53}}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_{53}})$$
$$\widetilde{\varphi_{54}} = (\widetilde{\gamma_{54}}\widetilde{\alpha_4} - \widetilde{\gamma_4}\widetilde{\alpha_{54}})(\widetilde{\beta_2}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_2}) - (\widetilde{\gamma_2}\widetilde{\alpha_4} - \widetilde{\gamma_4}\widetilde{\alpha_2})(\widetilde{\beta_{54}}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_{54}})$$

$$\widetilde{\lambda_{51}} = (\widetilde{\delta_{51}}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_{51}})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4}) - (\widetilde{\delta_4}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_4})(\widetilde{\beta_{51}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{51}})$$
$$\widetilde{\lambda_{52}} = (\widetilde{\delta_{52}}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_{52}})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4}) - (\widetilde{\delta_4}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_4})(\widetilde{\beta_{52}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{52}})$$
$$\widetilde{\lambda_{53}} = (\widetilde{\delta_{53}}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_{53}})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4}) - (\widetilde{\delta_4}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_4})(\widetilde{\beta_{53}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{53}})$$
$$\widetilde{\lambda_{54}} = (\widetilde{\delta_{54}}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_{54}})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4}) - (\widetilde{\delta_4}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_4})(\widetilde{\beta_{54}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{54}})$$

$$\widetilde{\theta_{51}} = (\widetilde{\gamma_{51}}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_{51}})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4}) - (\widetilde{\gamma_4}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_4})(\widetilde{\beta_{51}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{51}})$$
$$\widetilde{\theta_{52}} = (\widetilde{\gamma_{52}}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_{52}})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4}) - (\widetilde{\gamma_4}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_4})(\widetilde{\beta_{52}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{52}})$$
$$\widetilde{\theta_{53}} = (\widetilde{\gamma_{53}}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_{53}})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4}) - (\widetilde{\gamma_4}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_4})(\widetilde{\beta_{53}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{53}})$$
$$\widetilde{\theta_{54}} = (\widetilde{\gamma_{54}}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_{54}})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4}) - (\widetilde{\gamma_4}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_4})(\widetilde{\beta_{54}}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_{54}})$$

$$\widetilde{\rho_1} = (\widetilde{\delta_4}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_4})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2}) - (\widetilde{\delta_2}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_2})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4})$$
$$\widetilde{\rho_2} = (\widetilde{\gamma_3}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_3})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2}) - (\widetilde{\gamma_2}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_2})(\widetilde{\beta_3}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_3})$$
$$\widetilde{\rho_3} = (\widetilde{\delta_3}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_3})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2}) - (\widetilde{\delta_2}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_2})(\widetilde{\beta_3}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_3})$$
$$\widetilde{\rho_4} = (\widetilde{\gamma_4}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_4})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2}) - (\widetilde{\gamma_2}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_2})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4})$$
$$\widetilde{\rho_5} = (\widetilde{\delta_3}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_3})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4}) - (\widetilde{\delta_4}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_4})(\widetilde{\beta_3}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_3})$$
$$\widetilde{\rho_6} = (\widetilde{\gamma_2}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_2})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4}) - (\widetilde{\gamma_4}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_4})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2})$$
$$\widetilde{\rho_7} = (\widetilde{\delta_2}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_2})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4}) - (\widetilde{\delta_4}\widetilde{\alpha_1} - \widetilde{\delta_1}\widetilde{\alpha_4})(\widetilde{\beta_2}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_2})$$
$$\widetilde{\rho_8} = (\widetilde{\gamma_3}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_3})(\widetilde{\beta_4}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_4}) - (\widetilde{\gamma_4}\widetilde{\alpha_1} - \widetilde{\gamma_1}\widetilde{\alpha_4})(\widetilde{\beta_3}\widetilde{\alpha_1} - \widetilde{\beta_1}\widetilde{\alpha_3})$$
$$\widetilde{\rho_9} = (\widetilde{\delta_3}\widetilde{\alpha_4} - \widetilde{\delta_4}\widetilde{\alpha_3})(\widetilde{\beta_2}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_2}) - (\widetilde{\delta_2}\widetilde{\alpha_4} - \widetilde{\delta_4}\widetilde{\alpha_2})(\widetilde{\beta_3}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_3})$$
$$\widetilde{\rho_{10}} = (\widetilde{\gamma_1}\widetilde{\alpha_4} - \widetilde{\gamma_4}\widetilde{\alpha_1})(\widetilde{\beta_2}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_2}) - (\widetilde{\gamma_2}\widetilde{\alpha_4} - \widetilde{\gamma_4}\widetilde{\alpha_2})(\widetilde{\beta_1}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_1})$$
$$\widetilde{\rho_{11}} = (\widetilde{\delta_1}\widetilde{\alpha_4} - \widetilde{\delta_4}\widetilde{\alpha_1})(\widetilde{\beta_2}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_2}) - (\widetilde{\delta_2}\widetilde{\alpha_4} - \widetilde{\delta_4}\widetilde{\alpha_2})(\widetilde{\beta_1}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_1})$$
$$\widetilde{\rho_{12}} = (\widetilde{\gamma_3}\widetilde{\alpha_4} - \widetilde{\gamma_4}\widetilde{\alpha_3})(\widetilde{\beta_4}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_2}) - (\widetilde{\gamma_2}\widetilde{\alpha_4} - \widetilde{\gamma_4}\widetilde{\alpha_2})(\widetilde{\beta_3}\widetilde{\alpha_4} - \widetilde{\beta_4}\widetilde{\alpha_3})$$

$$\widetilde{\alpha_1} = \widetilde{\eta_{11}} + \widetilde{\kappa_{11}}\widetilde{\tau_{11}} + \widetilde{\psi_{11}}[\widetilde{\mu_{11}} - (\widetilde{\kappa_{11}}/4\eta_1)(b_s + \theta_s)^2] - 2(\delta_{12} - \delta_{11})^2(\widetilde{\gamma\vartheta_1} - \sigma)\gamma\widetilde{\varrho_1}$$
$$\widetilde{\alpha_2} = \widetilde{\eta_{12}} + \widetilde{\kappa_{11}}\widetilde{\tau_{12}} + \widetilde{\psi_{12}}[\widetilde{\mu_{11}} - (\widetilde{\kappa_{11}}/4\eta_1)(b_s + \theta_s)^2] - 2(\delta_{12} - \delta_{11})^2(\widetilde{\gamma\vartheta_2} - \sigma)\gamma\widetilde{\varrho_1}$$
$$\widetilde{\alpha_3} = 2\widetilde{\pi_{11}} + \widetilde{\kappa_{11}}\widetilde{\mu_{11}} + \widetilde{\kappa_{11}}[\widetilde{\mu_{11}} - (\widetilde{\kappa_{11}}/4\eta_1)(b_s + \theta_s)^2] - 2(\delta_{12} - \delta_{11})^2\gamma^2\widetilde{\varrho_1}^2$$
$$\widetilde{\alpha_4} = \widetilde{\pi_{12}} + \widetilde{\kappa_{11}}\widetilde{\mu_{12}} + \widetilde{\kappa_{12}}[\widetilde{\mu_{11}} - (\widetilde{\kappa_{11}}/4\eta_1)(b_s + \theta_s)^2] - 2(\delta_{12} - \delta_{11})^2\gamma^2\widetilde{\varrho_1}\widetilde{\varrho_2}$$
$$\widetilde{\alpha_{51}} = \widetilde{\upsilon_{11}} + \widetilde{\kappa_{11}}\widehat{\sigma_{11}} + \widehat{\alpha_{11}}[\widetilde{\mu_{11}} - (\widetilde{\kappa_{11}}/4\eta_1)(b_s + \theta_s)^2]$$
$$\widetilde{\alpha_{52}} = \widetilde{\upsilon_{12}} + \widetilde{\kappa_{11}}\widehat{\sigma_{12}} + \widehat{\alpha_{12}}[\widetilde{\mu_{11}} - (\widetilde{\kappa_{11}}/4\eta_1)(b_s + \theta_s)^2]$$
$$\widetilde{\alpha_{53}} = \widetilde{\omega_{11}} - \widehat{\pi_{11}} + \widetilde{\kappa_{11}}\widehat{\theta_{11}} + (\widehat{\omega_{11}} - 1)[\widetilde{\mu_{11}} - (\widetilde{\kappa_{11}}/4\eta_1)(b_s + \theta_s)^2]$$
$$\widetilde{\alpha_{54}} = \widetilde{\omega_{12}} + \widetilde{\kappa_{11}}\widehat{\theta_{12}} + \widehat{\omega_{12}}[\widetilde{\mu_{11}} - (\widetilde{\kappa_{11}}/4\eta_1)(b_s + \theta_s)^2]$$

$$\widetilde{\beta_1} = -\eta_1 + \widetilde{\psi_{11}}\widetilde{\tau_{11}} + \widetilde{\psi_{11}}[\widetilde{\tau_{11}} - (\widetilde{\psi_{11}}/4\eta_1)(b_s + \theta_s)^2] - 2(\delta_{12} - \delta_{11})^2(\widetilde{\gamma\vartheta_1} - \sigma)^2$$
$$\widetilde{\beta_2} = \widetilde{\psi_{11}}\widetilde{\tau_{12}} + \widetilde{\psi_{12}}[\widetilde{\tau_{11}} - (\widetilde{\psi_{11}}/4\eta_1)(b_s + \theta_s)^2] - 2(\delta_{12} - \delta_{11})^2(\widetilde{\gamma\vartheta_1} - \sigma)(\widetilde{\gamma\vartheta_2} - \sigma)$$
$$\widetilde{\beta_3} = \widetilde{\eta_{11}} + \widetilde{\psi_{11}}\widetilde{\mu_{11}} + \widetilde{\kappa_{11}}[\widetilde{\tau_{11}} - (\widetilde{\psi_{11}}/4\eta_1)(b_s + \theta_s)^2] - 2(\delta_{12} - \delta_{11})^2(\widetilde{\gamma\vartheta_1} - \sigma)\gamma\widetilde{\varrho_1}$$
$$\widetilde{\beta_4} = \widetilde{\psi_{11}}\widetilde{\mu_{12}} + \widetilde{\kappa_{12}}[\widetilde{\tau_{11}} - (\widetilde{\psi_{11}}/4\eta_1)(b_s + \theta_s)^2] - 2(\delta_{12} - \delta_{11})^2(\widetilde{\gamma\vartheta_1} - \sigma)\gamma\widetilde{\varrho_2}$$
$$\widetilde{\beta_{51}} = \widetilde{\psi_{11}}\widehat{\sigma_{11}} + \widehat{\alpha_{11}}[\widetilde{\tau 11} - (\widetilde{\psi_{11}}/4\eta_1)(b_s + \theta_s)^2]$$
$$\widetilde{\beta_{52}} = \widetilde{\psi_{11}}\widehat{\sigma_{12}} + \widehat{\alpha_{12}}[\widetilde{\tau_{11}} - (\widetilde{\psi_{11}}/4\eta_1)(b_s + \theta_s)^2]$$
$$\widetilde{\beta_{53}} = -\widehat{\eta_{11}} + \widetilde{\psi_{11}}\widehat{\theta_{11}} + (\widehat{\omega_{11}} - 1)[\widetilde{\tau_{11}} - (\widetilde{\psi_{11}}/4\eta_1)(b_s + \theta_s)^2]$$
$$\widetilde{\beta_{54}} = \widetilde{\psi_{11}}\widehat{\theta_{12}} + \widehat{\omega_{12}}[\widetilde{\tau_{11}} - (\widetilde{\psi_{11}}/4\eta_1)(b_s + \theta_s)^2]$$

$$\widetilde{\gamma_1} = \widetilde{\eta_{21}} + \widetilde{\kappa_{22}}\widetilde{\tau_{21}} + \widetilde{\psi_{21}}[\widetilde{\mu_{22}} - (\widetilde{\kappa_{22}}/4\eta_2)(b_s + \theta_s)^2] - 2(\delta_{22} - \delta_{21})^2(\widetilde{\gamma\vartheta_1} - \sigma)\gamma\widetilde{\varrho_2}$$
$$\widetilde{\gamma_2} = \widetilde{\eta_{22}} + \widetilde{\kappa_{22}}\widetilde{\tau_{22}} + \widetilde{\psi_{22}}[\widetilde{\mu_{22}} - (\widetilde{\kappa_{22}}/4\eta_2)(b_s + \theta_s)^2] - 2(\delta_{22} - \delta_{21})^2(\widetilde{\gamma\vartheta_2} - \sigma)\gamma\widetilde{\varrho_2}$$
$$\widetilde{\gamma_3} = \widetilde{\pi_{21}} + \widetilde{\kappa_{22}}\widetilde{\mu_{21}} + \widetilde{\kappa_{21}}[\widetilde{\mu_{22}} - (\widetilde{\kappa_{22}}/4\eta_2)(b_s + \theta_s)^2] - 2(\delta_{22} - \delta_{21})^2\gamma^2\widetilde{\varrho_1}\widetilde{\varrho_2}$$
$$\widetilde{\gamma_4} = 2\widetilde{\pi_{22}} + \widetilde{\kappa_{22}}\widetilde{\mu_{22}} + \widetilde{\kappa_{22}}[\widetilde{\mu_{22}} - (\widetilde{\kappa_{22}}/4\eta_2)(b_s + \theta_s)^2] - 2(\delta_{22} - \delta_{21})^2\gamma^2\widetilde{\varrho_2}^2$$
$$\widetilde{\gamma_{51}} = \widetilde{\upsilon_{21}} + \widetilde{\kappa_{22}}\widehat{\sigma_{21}} + \widehat{\alpha_{21}}[\widetilde{\mu_{22}} - (\widetilde{\kappa_{22}}/4\eta_2)(b_s + \theta_s)^2]$$
$$\widetilde{\gamma_{52}} = \widetilde{\upsilon_{22}} + \widetilde{\kappa_{22}}\widehat{\sigma_{22}} + \widehat{\alpha_{22}}[\widetilde{\mu_{22}} - (\widetilde{\kappa_{22}}/4\eta_2)(b_s + \theta_s)^2]$$
$$\widetilde{\gamma_{53}} = \widetilde{\omega_{21}} + \widetilde{\kappa_{22}}\widehat{\theta_{21}} + \widehat{\omega_{21}}[\widetilde{\mu_{22}} - (\widetilde{\kappa_{22}}/4\eta_2)(b_s + \theta_s)^2]$$
$$\widetilde{\gamma_{54}} = \widetilde{\omega_{22}} - \widehat{\pi_{22}} + \widetilde{\kappa_{22}}\widehat{\theta_{22}} + (\widehat{\omega_{22}} - 1)[\widetilde{\mu_{22}} - (\widetilde{\kappa_{22}}/4\eta_2)(b_s + \theta_s)^2]$$

$$\widetilde{\delta_1} = \widetilde{\psi_{22}}\widetilde{\tau_{21}} + \widetilde{\psi_{21}}[\widetilde{\tau_{22}} - (\widetilde{\kappa_{22}}/4\eta_2)(b_s+\theta_s)^2] - 2(\delta_{22}-\delta_{21})^2(\gamma\widetilde{\vartheta_1} - \sigma)(\gamma\widetilde{\vartheta_2} - \sigma)$$
$$\widetilde{\delta_2} = -\eta_1 + \widetilde{\psi_{22}}\widetilde{\tau_{22}} + \widetilde{\psi_{22}}[\widetilde{\tau_{22}} - (\widetilde{\kappa_{22}}/4\eta_2)(b_s+\theta_s)^2] - 2(\delta_{22}-\delta_{21})^2(\gamma\widetilde{\vartheta_2} - \sigma)^2$$
$$\widetilde{\delta_3} = \widetilde{\psi_{22}}\widetilde{\mu_{21}} + \widetilde{\kappa_{21}}[\widetilde{\tau_{22}} - (\widetilde{\kappa_{22}}/4\eta_2)(b_s+\theta_s)^2] - 2(\delta_{22}-\delta_{21})^2(\gamma\widetilde{\vartheta_2} + \sigma)\gamma\tilde{\varrho}_1$$
$$\widetilde{\delta_4} = \widetilde{\eta_{22}} + \widetilde{\psi_{22}}\widetilde{\mu_{22}} + \widetilde{\kappa_{22}}[\widetilde{\tau_{22}} - (\widetilde{\kappa_{22}}/4\eta_2)(b_s+\theta_s)^2] - 2(\delta_{22}-\delta_{21})^2(\gamma\widetilde{\vartheta_2} + \sigma)\gamma\tilde{\varrho}_2$$
$$\widetilde{\delta_{51}} = \widetilde{\psi_{22}}\widehat{\sigma_{21}} + \widehat{\alpha_{21}}[\widetilde{\tau 22} - (\widetilde{\psi_{22}}/4\eta_2)(b_s+\theta_s)^2]$$
$$\widetilde{\delta_{52}} = \widetilde{\psi_{22}}\widehat{\sigma_{22}} + \widehat{\alpha_{22}}[\widetilde{\tau_{22}} - (\widetilde{\psi_{22}}/4\eta_2)(b_s+\theta_s)^2]$$
$$\widetilde{\delta_{53}} = \widetilde{\psi_{22}}\widehat{\theta_{21}} + \widehat{\omega_{21}}[\widetilde{\tau_{22}} - (\widetilde{\psi_{22}}/4\eta_2)(b_s+\theta_s)^2]$$
$$\widetilde{\delta_{54}} = -\widehat{\eta_{22}} + \widetilde{\psi_{22}}\widehat{\theta_{22}} + (\widehat{\omega_{22}} - 1)[\widetilde{\tau_{22}} - (\widetilde{\psi_{22}}/4\eta_2)(b_s+\theta_s)^2]$$

$$\widetilde{\eta_{11}} = -(b_p+\theta_p)\widehat{\nu}(\widehat{\vartheta_3}\widehat{\nu_2} - \widehat{\vartheta_2}\widehat{\nu_3}) - \theta_p\widehat{\nu}(\widehat{\vartheta_3}\widehat{\nu_1} - \widehat{\vartheta_1}\widehat{\nu_3}) + (b_s+\theta_s)$$
$$\widetilde{\eta_{12}} = -(b_p+\theta_p)\widehat{\nu}(\widehat{\vartheta_4}\widehat{\nu_2} - \widehat{\vartheta_2}\widehat{\nu_4}) - \theta_p\widehat{\nu}(\widehat{\vartheta_4}\widehat{\nu_1} - \widehat{\vartheta_1}\widehat{\nu_4}) - \theta_s$$
$$\widetilde{\eta_{21}} = \theta_p\widehat{\nu}(\widehat{\vartheta_3}\widehat{\nu_2} - \widehat{\vartheta_2}\widehat{\nu_3}) + (b_p+\theta_p)\widehat{\nu}(\widehat{\vartheta_3}\widehat{\nu_1} - \widehat{\vartheta_1}\widehat{\nu_3}) - \theta_s$$
$$\widetilde{\eta_{22}} = \theta_p\widehat{\nu}(\widehat{\vartheta_4}\widehat{\nu_2} - \widehat{\vartheta_2}\widehat{\nu_4}) + (b_p+\theta_p)\widehat{\nu}(\widehat{\vartheta_4}\widehat{\nu_1} - \widehat{\vartheta_1}\widehat{\nu_4}) + (b_s+\theta_s)$$
$$\widetilde{\pi_{11}} = (b_p+\theta_p)\widehat{\nu}(\theta_p\widehat{\nu_2} + \widehat{\vartheta_2}(b_p+\theta_p)) + \theta_p\widehat{\nu}(\theta_p\widehat{\nu_1} + \widehat{\vartheta_1}(b_p+\theta_p))$$
$$\widetilde{\pi_{12}} = -(b_p+\theta_p)\widehat{\nu}((b_p+\theta_p)\widehat{\nu_2} + \widehat{\vartheta_2}\theta_p) - \theta_p\widehat{\nu}((b_p+\theta_p)\widehat{\nu_1} + \widehat{\vartheta_1}\theta_p)$$
$$\widetilde{\pi_{21}} = -\theta_p\widehat{\nu}(\theta_p\widehat{\nu_2} + \widehat{\vartheta_2}(b_p+\theta_p)) - (b_p+\theta_p)\widehat{\nu}(\theta_p\widehat{\nu_1} + \widehat{\vartheta_1}(b_p+\theta_p))$$
$$\widetilde{\pi_{22}} = \theta_p\widehat{\nu}((b_p+\theta_p)\widehat{\nu_2} + \widehat{\vartheta_2}\theta_p) + (b_p+\theta_p)\widehat{\nu}((b_p+\theta_p)\widehat{\nu_1} + \widehat{\vartheta_1}\theta_p)$$

$$\widetilde{\chi_{11}} = \varphi_1\gamma(\widehat{\alpha_1} + D_2\widehat{\alpha_2}) = -\widetilde{\chi_{12}}$$
$$\widetilde{\chi_{21}} = \varphi_2\gamma(D_1\widehat{\alpha_1} + \widehat{\alpha_2}) = -\widetilde{\chi_{22}}$$
$$\widetilde{\varpi_{11}} = \varphi_1\sigma(\widehat{\alpha_1} + D_2\widehat{\alpha_2}) = -\widetilde{\varpi_{12}}$$
$$\widetilde{\varpi_{21}} = \varphi_2\sigma(D_1\widehat{\alpha_1} + \widehat{\alpha_2}) = -\widetilde{\varpi_{22}}$$

$$\widetilde{\psi_{11}} = \widetilde{\chi_{11}}\widehat{\nu}(\widehat{\vartheta_3}\widehat{\nu_2} - \widehat{\vartheta_2}\widehat{\nu_3}) - \widetilde{\chi_{12}}\widehat{\nu}(\widehat{\vartheta_3}\widehat{\nu_1} - \widehat{\vartheta_1}\widehat{\nu_3}) + \widetilde{\varpi_{11}}$$
$$\widetilde{\psi_{12}} = \widetilde{\chi_{11}}\widehat{\nu}(\widehat{\vartheta_4}\widehat{\nu_2} - \widehat{\vartheta_2}\widehat{\nu_4}) - \widetilde{\chi_{12}}\widehat{\nu}(\widehat{\vartheta_4}\widehat{\nu_1} - \widehat{\vartheta_1}\widehat{\nu_4}) + \widetilde{\varpi_{12}}$$
$$\widetilde{\psi_{21}} = \widetilde{\chi_{21}}\widehat{\nu}(\widehat{\vartheta_3}\widehat{\nu_2} - \widehat{\vartheta_2}\widehat{\nu_3}) - \widetilde{\chi_{22}}\widehat{\nu}(\widehat{\vartheta_3}\widehat{\nu_1} - \widehat{\vartheta_1}\widehat{\nu_3}) + \widetilde{\varpi_{21}}$$
$$\widetilde{\psi_{22}} = \widetilde{\chi_{21}}\widehat{\nu}(\widehat{\vartheta_4}\widehat{\nu_2} - \widehat{\vartheta_2}\widehat{\nu_4}) - \widetilde{\chi_{22}}\widehat{\nu}(\widehat{\vartheta_4}\widehat{\nu_1} - \widehat{\vartheta_1}\widehat{\nu_4}) + \widetilde{\varpi_{22}}$$

$$
\begin{aligned}
\widetilde{\kappa_{11}} &= -\widetilde{\chi_{11}}\widehat{\nu}(\theta_p\widehat{\nu_2} + \widehat{\vartheta_2}(b_p+\theta_p)) + \widetilde{\chi_{12}}\widehat{\nu}(\theta_p\widehat{\nu_1} + \widehat{\vartheta_1}(b_p+\theta_p)) \\
\widetilde{\kappa_{12}} &= \widetilde{\chi_{11}}\widehat{\nu}((b_p+\theta_p)\widehat{\nu_2} + \widehat{\vartheta_2}\theta_p) - \widetilde{\chi_{12}}\widehat{\nu}((b_p+\theta_p)\widehat{\nu_1} + \widehat{\vartheta_1}\theta_p) \\
\widetilde{\kappa_{21}} &= -\widetilde{\chi_{21}}\widehat{\nu}(\theta_p\widehat{\nu_2} + \widehat{\vartheta_2}(b_p+\theta_p)) - \widetilde{\chi_{22}}\widehat{\nu}(\theta_p\widehat{\nu_1} + \widehat{\vartheta_1}(b_p+\theta_p)) \\
\widetilde{\kappa_{22}} &= \widetilde{\chi_{21}}\widehat{\nu}((b_p+\theta_p)\widehat{\nu_2} + \widehat{\vartheta_2}\theta_p) + \widetilde{\chi_{22}}\widehat{\nu}((b_p+\theta_p)\widehat{\nu_1} + \widehat{\vartheta_1}\theta_p) \\
\hline
\widetilde{\tau_{11}} &= \widehat{\kappa_3}\widehat{\nu}(\widehat{\vartheta_3}\widehat{\nu_2} - \widehat{\vartheta_2}\widehat{\nu_3}) + \widehat{\kappa_3}\widehat{\nu}(\widehat{\vartheta_3}\widehat{\nu_1} - \widehat{\vartheta_1}\widehat{\nu_3}) + \widehat{\mu_3} \\
\widetilde{\tau_{12}} &= \widehat{\kappa_3}\widehat{\nu}(\widehat{\vartheta_4}\widehat{\nu_2} - \widehat{\vartheta_2}\widehat{\nu_4}) + \widehat{\kappa_3}\widehat{\nu}(\widehat{\vartheta_4}\widehat{\nu_1} - \widehat{\vartheta_1}\widehat{\nu_4}) - \widehat{\mu_3} \\
\widetilde{\tau_{21}} &= \widehat{\kappa_6}\widehat{\nu}(\widehat{\vartheta_3}\widehat{\nu_2} - \widehat{\vartheta_2}\widehat{\nu_3}) + \widehat{\kappa_6}\widehat{\nu}(\widehat{\vartheta_3}\widehat{\nu_1} - \widehat{\vartheta_1}\widehat{\nu_3}) + \widehat{\mu_6} \\
\widetilde{\tau_{22}} &= \widehat{\kappa_6}\widehat{\nu}(\widehat{\vartheta_4}\widehat{\nu_2} - \widehat{\vartheta_2}\widehat{\nu_4}) + \widehat{\kappa_6}\widehat{\nu}(\widehat{\vartheta_4}\widehat{\nu_1} - \widehat{\vartheta_1}\widehat{\nu_4}) - \widehat{\mu_6} \\
\hline
\widetilde{\mu_{11}} &= -\widehat{\kappa_3}\widehat{\nu}(\theta_p\widehat{\nu_2} + \widehat{\vartheta_2}(b_p+\theta_p)) + \widehat{\kappa_3}\widehat{\nu}(\theta_p\widehat{\nu_1} + \widehat{\vartheta_1}(b_p+\theta_p)) \\
\widetilde{\mu_{12}} &= \widehat{\kappa_3}\widehat{\nu}((b_p+\theta_p)\widehat{\nu_2} + \widehat{\vartheta_2}\theta_p) - \widehat{\kappa_3}\widehat{\nu}((b_p+\theta_p)\widehat{\nu_1} + \widehat{\vartheta_1}\theta_p) \\
\widetilde{\mu_{21}} &= -\widehat{\kappa_6}\widehat{\nu}(\theta_p\widehat{\nu_2} + \widehat{\vartheta_2}(b_p+\theta_p)) - \widehat{\kappa_6}\widehat{\nu}(\theta_p\widehat{\nu_1} + \widehat{\vartheta_1}(b_p+\theta_p)) \\
\widetilde{\mu_{22}} &= \widehat{\kappa_6}\widehat{\nu}((b_p+\theta_p)\widehat{\nu_2} + \widehat{\vartheta_2}\theta_p) + \widehat{\kappa_6}\widehat{\nu}((b_p+\theta_p)\widehat{\nu_1} + \widehat{\vartheta_1}\theta_p) \\
\hline
\widetilde{\vartheta_1} &= 2b_p\widehat{\nu}(\widehat{\nu_3} - \widehat{\vartheta_3}), \quad \widetilde{\vartheta_2} = 2b_p\widehat{\nu}(\widehat{\nu_4} - \widehat{\vartheta_4}) \\
\widetilde{\vartheta_3} &= 2b_p\widehat{\nu}(\widehat{\nu_5} - \widehat{\vartheta_5}), \quad \widetilde{\varrho_1} = -\widetilde{\varrho_2} = 2b_p\widehat{\nu}(b_p + 2\theta_p)
\end{aligned}
$$

Appendix E. Proof to Theorem 5.2

The system given in Equation 31 is

$$\mathbf{\Phi(t) = M\Phi(t-1)}.$$

Let $\mathbf{P}$ be the modal matrix of $\mathbf{M}$. That is, $\mathbf{P}$ is the 4x4 matrix whose 4 columns are the eigenvectors of $\mathbf{M}$. For a given $\mathbf{\Phi(t)}$, we define a vector $\mathbf{z(t)}$ by

$$\mathbf{\Phi(t) = Pz(t)}.$$

This transformation follows from the fact that any vector $\mathbf{\Phi(t)}$ can be written as a linear combination of its eigenvectors. That is $\mathbf{\Phi(t)}$ can be expressed as

$$\mathbf{\Phi(t) = z_1(t)e_1} + \mathrm{z}_2(\mathrm{t})\mathbf{e_2} + \mathrm{z}_3(\mathrm{t})\mathbf{e_3} + \mathrm{z}_4(\mathrm{t})\mathbf{e_4}.$$

where $z_i(t), i \in 1, 2, 3, 4$ are scalars. Using the fact that $\mathbf{Me_i} = \lambda_\mathbf{i}\mathbf{e_i}$ (where λ_i is an eigenvalue of $\mathbf{M}$), multiplying the equation above by matrix $\mathbf{M}$ yields

$$
\begin{aligned}
\mathbf{\Phi(t+1)} &= \mathbf{M\Phi(t)} && (34) \\
&= \lambda_1 z_1(t)\mathbf{e_1} + \lambda_2\mathrm{z}_2(\mathrm{t})\mathbf{e_2} + \lambda_3\mathrm{z}_3(\mathrm{t})\mathbf{e_3} + \lambda_4\mathrm{z}_4(\mathrm{t})\mathbf{e_4}. && (35)
\end{aligned}
$$

In this new transformation, the original system in Equation 31 can be represented as

$$\mathbf{Pz(t+1)} = \mathbf{MPz(t)}.$$

or, equivalently,

$$\mathbf{z(t+1)} = \mathbf{P}^{-1}\mathbf{MPz(t)}. \tag{36}$$

This defines a new system that is related to the original system by a change of variable. The new system matrix $\mathbf{P}^{-1}\mathbf{MP}$ is equal to $\mathbf{\Lambda}$, where $\mathbf{\Lambda}$ is the diagonal matrix with the eigenvalues of $\mathbf{M}$ on the diagonal. Thus, when written out in detail, Equation 36 becomes

$$\begin{bmatrix} z_1(t+1) \\ z_2(t+1) \\ z_3(t+1) \\ z_4(t+1) \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 & 0 & 0 \\ 0 & \lambda_2 & 0 & 0 \\ 0 & 0 & \lambda_3 & 0 \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix} \begin{bmatrix} z_1(t) \\ z_2(t) \\ z_3(t) \\ z_4(t) \end{bmatrix} \tag{37}$$

which explicitly displays the diagonal form obtained by the change of variable.

The state-transition matrix of a constant coefficient discrete-time system at period k is $\mathbf{M}^k$. The system matrix can be calculated by first converting $\mathbf{M}$ to diagonal form as

$$\mathbf{M} = \mathbf{P\Lambda P}^{-1}.$$

which provides a representation of $\mathbf{M}$ in terms of its eigenvalues and eigenvectors. It then follows that for any

$k \geq 0$

$$\mathbf{M^k} = \mathbf{P\Lambda^k P}^{-1}.$$

Therefore, calculation of $\mathbf{M^k}$ is transferred to the calculation of Λ^k. Since Λ is diagonal, one finds immediately that

$$\Lambda^k = \begin{bmatrix} \lambda_1^k & 0 & 0 & 0 \\ 0 & \lambda_2^k & 0 & 0 \\ 0 & 0 & \lambda_3^k & 0 \\ 0 & 0 & 0 & \lambda_4^k \end{bmatrix} \tag{38}$$

One can see immediately that if the magnitude of dominant eigenvalue is less than 1 (i.e., $|\lambda_i| < 1$ for all i), as k increases, Λ^k tends toward zero. This corresponds to the system converging over time. On the other hand, if there is at least one eigenvalue with magnitude greater than one ($|\lambda_j| > 1$ for some j), the system of market bases evolution will increases geometrically towards infinity. This corresponds to a divergent system.

Appendix F. Proof to Theorem 5.3

Let λ be the dominant eigenvalue of a discrete time system. It is possible to express λ in the form

$$\lambda = re^{i\theta} = r(\cos\theta + i\sin\theta).$$

The characteristic response due to this eigenvalue is

$$\lambda^k = r^k e^{ik\theta} = r^k(\cos k\theta + i\sin k\theta).$$

The coefficient that multiplies the associated eigenvector varies according to this characteristic pattern. From the above equation, one can see that if λ is real and positive, the response pattern is the geometric sequence r^k, which increases if $r > 1$ and decreases if $r < 1$. No oscillation will occur with positive eigenvalue since r^k remains positive for any k. However, if λ is negative, the response will be an alternating geometric sequence since r^k switches sign for every step.

If λ is complex, it will appear with its complex conjugate. The real response due to both eigenvalues is of the form $r^k(A\cos k\theta + iB\sin k\theta)$. If $\theta \neq 0$, the expression within the parentheses will change sign a k changes. However, the exact pattern of variation will not be perfectly regular. In our problem, we assume that λ is not complex. Therefore, this irregular oscillation case is excluded from our analysis.

Appendix G. Range of Parameters Used in Numerical Studies

Parameter	Range
b_p	{0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5}
θ_p	{0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5}
b_s	{0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5}
θ_s	{0.3, 0.5, 0.7, 0.9, 1.1, 1.3, 1.5}
a_i	{40, 60, 80, 100, 120}
c_i	{2, 4, 6, 8, 10}
η_i	{2, 4, 6, 8, 10}
γ	{0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0}
σ	{0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0}
β	{0.5, 1.0, 1.5, 2.0, 2.5, 3.0, 3.5, 4.0}

The range of these parameters are based on related literature such as Tsay and Agrawal (2000) [24] and Vilcassin *et al.*(1999) [25].

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